Sheaves of ϕ-Principal Parts

by

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Abstract

This thesis studies sheaves of ϕ-principal parts $\mathcal{P}^r_\phi(\mathcal{F})$, a modification of the sequence $\mathcal{P}^r(\mathcal{F})$ of sheaves of r-th principal parts. After constructing the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$ and undertaking their general study in Chapter 1, we apply the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$ to the problem of counting how many Weierstrass points (counted according to their Weierstrass weight) are absorbed into a singularity in a curve degeneration in Chapter 2, and to the problem of counting lines in a linear system in a Grassmannian in Chapter 3.

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Statement of Originality

I hereby certify that all of the work described within this thesis is the original work of the author. Any published (or unpublished) ideas and/or techniques from the work of others are fully acknowledged in accordance with the standard referencing practices.

Contents

List of Figures

Chapter 1

Introduction

Let X be an algebraic variety (or, more generally, a scheme of finite type) over an algebraically closed field k, M a line bundle over X, $\phi\colon \Omega_X \to M$ an $\mathcal{O}_X$-linear morphism, and $\mathcal{F}$ an $\mathcal{O}_X$-module.

The contents of this thesis are situated in the circle of ideas around the sequence $\mathcal{P}^r_\phi(\mathcal{F})$ of sheaves over X (these sheaves are indexed by integers $r \geq 0$, and depend on M, ϕ and $\mathcal{F}$ in a manner that will be described below). In Chapter 2, the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$ are constructed and several of their general properties are established, in Chapter 3, the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$ are applied to the problem of computing how much Weierstrass weight gets absorbed into a singular point in a curve degeneration, and in Chapter 4, the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$ are applied to the problem of counting the number of lines that appear in a linear system in a Grassmannian.

The sequence $\mathcal{P}^r_\phi(\mathcal{F})$ is a variant (in an informal sense) of the sequence of sheaves of principal parts $\mathcal{P}^r(\mathcal{F})$; the latter is a well-known instrument in algebraic geometry.[1] To help explain the analogy between $\mathcal{P}^r_\phi(\mathcal{F})$ and $\mathcal{P}^r(\mathcal{F})$, we begin by reminding the reader of the sheaves $\mathcal{P}^r(\mathcal{F})$ and the way these sheaves are applied. Then, we introduce the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$, and summarize the contents of the individual chapters of this thesis.

[1] The sheaf $\mathcal{P}^r(\mathcal{F})$ is sometimes called the r-th jet sheaf of L, but we follow [Gro67] and [EH16] in calling $\mathcal{P}^r(L)$ the r-th sheaf of principal parts (in part to avoid confusion with the scheme $\mathrm{Hom}(\operatorname{Spec} k[x]/(x^{r+1}), X)$ that is usually called the scheme of r-jets of X).

1.1 Sheaves of principal parts $\mathcal{P}^r(\mathcal{F})$ in algebraic geometry

The sheaf $\mathcal{P}^r(\mathcal{F})$ of r-th order principal parts of $\mathcal{F}$ comes equipped with a differential operator $d^r : \mathcal{F} \to \mathcal{P}^r(\mathcal{F})$ of order r. The pair $(\mathcal{P}^r(\mathcal{F}), d^r)$ is the object (unique up to isomorphism) satisfying the following universal property: for any differential operator $D\colon \mathcal{F} \to \mathcal{E}$ of order at most r, there is a unique induced $\mathcal{O}_X$-linear morphism $\mathcal{P}^r(\mathcal{F}) \to \mathcal{E}$ that makes the triangle

$$\begin{tikzcd} \mathcal{P}^r(\mathcal{F}) \arrow[r, "\exists!"] & \mathcal{E} \\ \mathcal{F} \arrow[u, "d^r"] \arrow[ur, "D"'] & \end{tikzcd}$$

commute.

One possible construction of $\mathcal{P}^r(\mathcal{F})$ (cf. [Gro67, §16.3]) begins with the special case $\mathcal{P}^r(\mathcal{O}_X) =: \mathcal{P}^r$. Let $\Delta\colon X \to X \times X$ denote the inclusion of the diagonal into $X \times X$. One defines

$$\mathcal{P}^r = \Delta^* \left(\mathcal{O}_{X\times X}/\mathcal{I}_\Delta^{r+1}\right),$$

which is clearly a sheaf of rings. The projections p_1 and p_2 of $X \times X$ onto its first and second factor, respectively, induce two morphisms $\mathcal{O}_X \to \mathcal{P}^r$ of sheaves of rings, which therefore induce two $\mathcal{O}_X$-module structures on $\mathcal{P}^r$. By convention, $\mathcal{P}^r$ is studied with its p_1-induced $\mathcal{O}_X$-module structure, and, with respect to this structure, p_2 is a differential operator of order r.

The sheaf $\mathcal{P}^r(\mathcal{F})$ is then defined by

$$\mathcal{P}^r(\mathcal{F}) = \mathcal{P}^r\ {}_2\!\otimes \mathcal{F},$$

where the symbol ${}_2\!\otimes$ indicates that the $\mathcal{P}^r$ on the left side is taken with its p_2-induced $\mathcal{O}_X$-module structure. The differential operator $\mathcal{F} \to \mathcal{P}^r(\mathcal{F})$ is then defined by the formula $f \mapsto 1_{\mathcal{P}^r} \otimes f$, where $1_{\mathcal{P}^r}$ is the multiplicative identity with respect to the ring structure of $\mathcal{P}^r$, and f is a local section of $\mathcal{F}$.

In applying $\mathcal{P}^r(\mathcal{F})$, $\mathcal{F}$ is often taken to be a line bundle L over X, and X is frequently assumed to be smooth. With these hypotheses, for all $r \geq 1$, there is a very useful short exact sequence

$$0 \longrightarrow \operatorname{Sym}^r(\Omega_X) \otimes L \longrightarrow \mathcal{P}^r(L) \xrightarrow{\pi_r} \mathcal{P}^{r-1}(L) \longrightarrow 0, \tag{1.1}$$

and as a first application of (1.1), we deduce that $\mathcal{P}^r(L)$ is a vector bundle when X is smooth.

Locally, the differential operator $L \to \mathcal{P}^r(L)$ may be thought of as the Taylor expansions of the sections of L up to (and including) order r. The projection map $\mathcal{P}^r(L) \xrightarrow{\pi_r} \mathcal{P}^{r-1}(L)$ is locally a projection map that omits the r-th order term of the Taylor expansion.

For a singular space X, the right-exact sequence

$$\operatorname{Sym}^r(\Omega_X) \otimes L \longrightarrow \mathcal{P}^r(L) \xrightarrow{\pi_r} \mathcal{P}^{r-1}(L) \longrightarrow 0$$

is in general not left-exact, and the sheaf $\mathcal{P}^r(L)$ is no longer locally free. This can make $\mathcal{P}^r(L)$ more difficult to work with in cases when X is singular.

We now turn to illustrating a typical use of the sheaves $\mathcal{P}^r(\mathcal{F})$ in intersection theory. Consider the following problem: given a general one-parameter family of degree d curves in $\mathbb{P}^2$, how many of the curves in the family are singular?

Before introducing the method of solving the above problem that applies the sheaves $\mathcal{P}^r(\mathcal{F})$, to get a feeling for the problem, we first consider the family of conic curves in $\mathbb{P}^2_{[X:Y:Z]}$ given by the equations

$$C_t \colon \left\{ \; [X:Y:Z] \in \mathbb{P}^2 : X^2 + tY^2 = (X + tY)Z \; \right\}$$

where $t \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ (and C_∞ is given by the equation $Y^2 = YZ$). How many of the curves in the family are singular?

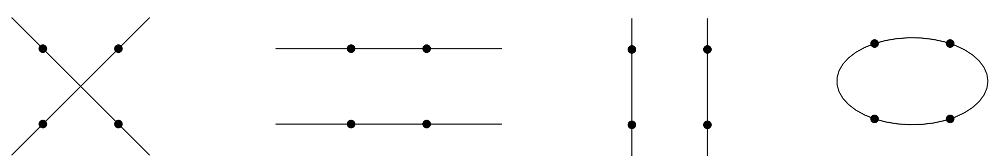
For this example, it is not difficult to compute the answer directly — a point $[X:Y:Z]$

of C_t is singular if and only if the 1×3 matrix $\begin{bmatrix} \partial F_t/\partial X & \partial F_t/\partial Y & \partial F_t/\partial Z \end{bmatrix}$ has rank zero at $[X : Y : Z]$ (F_t is the equation of C_t), and one checks quickly that the only values of t for which the resulting system of equations has solutions (that lie on C_t) are $t = -1, 0, \infty$. Therefore, exactly three of the conics are singular.

The problem can be solved by another method, which is attractively geometric, and which can be generalized to other one-parameter families of conics. We proceed as follows. One can check that the intersection of the curves C_t (as t runs over $\mathbb{P}^1$) consists of four points:

$$p_1 = [0 : 0 : 1], \qquad p_2 = [1 : 0 : 1], \qquad p_3 = [0 : 1 : 1], \qquad p_4 = [1 : 1 : 1],$$

which are the base points of the family $\{C_t\}_{t \in \mathbb{P}^1}$. Since the curves C_t are reduced, and no three of p_1, p_2, p_3, p_4 lie on a single line, the singular elements of the family $\{C_t\}$ are pairs of lines, each of which passes through two of the base points. Therefore, the number of singular curves in the family $\{C_t\}$ is equal to the number of pairs of disjoint two-element subsets of a four-element set, which is $\binom{4}{2}/2 = 3$.



$t = -1$ $\qquad$ $t = 0$ $\qquad$ $t = \infty$ $\qquad$ $t \neq -1, 0, \infty$

Figure 1.1: The (real points of the) three singular curves in the family $V(X^2 + tY^2 = (X + tY)Z)$ in the chart $Z \neq 0$ of $\mathbb{P}^2$ (the three figures starting from the far left and going right), as well as a typical smooth member of this family (on the far right). The four marked points are p_1, p_2, p_3, p_4.

It can be shown that a general one-parameter family of planar conics has exactly four base points (and conversely, there is a pencil of conics passing through any four points in general position); therefore, by the same argument as above, a general one-parameter family of planar conics contains three singular curves.

We now return to the problem for a family of curves of arbitrary degree. Instead of looking for analogues of the methods above, we translate the question into a Chern class computation by using bundles of principal parts. This proceeds as follows.

A global section of $\mathcal{O}(d)$ induces a global section of $\mathcal{P}^1(\mathcal{O}(d))$; writing F in local coordinates as $F = \sum_{0\le i+j\le d} a_{ij}\, x^i y^j$, the induced section is given by $F' = (F,\ \partial F/\partial x,\ \partial F/\partial y)$. We see that a point p is a singular point of the curve $F = 0$ if and only if F' vanishes at p. Therefore, the singular points of the family $\{\lambda\, F + \mu\, G = 0\}_{[\lambda:\mu]\in\mathbb{P}^1}$ correspond to the points where the linear combination $\lambda\, F' + \mu\, G'$ vanishes. The (rational equivalence class of the) locus of the latter is $c_2(\mathcal{P}^1(\mathcal{O}(d)))$ (conditional on F' and G' being general).

The short exact sequence (1.1) specializes to the short exact sequence

$$0 \to \Omega_{\mathbb{P}^2} \otimes \mathcal{O}(d) \to \mathcal{P}^1(\mathcal{O}(d)) \to \mathcal{O}(d) \to 0.$$

Therefore, $c(\mathcal{P}^1(\mathcal{O}(d))) = c(\Omega_{\mathbb{P}^2} \otimes \mathcal{O}(d))\, c(\mathcal{O}(d))$. On the other hand, tensoring the Euler sequence

$$0 \to \Omega_{\mathbb{P}^2} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2} \to 0$$

by $\mathcal{O}(d)$, we find that $c(\mathcal{O}(d-1)^{\oplus 3}) = c(\Omega_{\mathbb{P}^2} \otimes \mathcal{O}(d))c(\mathcal{O}(d))$, so that

$$c(\mathcal{P}^1(\mathcal{O}(d))) = c(\mathcal{O}(d-1)^{\oplus 3}) = (1 + (d-1)H)^3 = 1 + 3(d-1)H + 3(d-1)^2 H^2.$$

The problem is solved: a general one-parameter family of degree d curves contains $3(d-1)^2$ singular curves (we mention but do not elaborate on a slight subtlety — to reach the above conclusion, it is necessary to show that each member has at most one singular point).

1.2 Thesis contents

Chapter 2. The sheaves $\mathcal{P}^r_\phi(\mathcal{F})$. Continuing with the above notation, let X be a scheme of finite type over an algebraically closed field k, M a line bundle on X, $\phi\colon \Omega_X \to M$ an $\mathcal{O}_X$-linear morphism, $D_\phi = \phi \circ d$ (or simply D) the derivation associated to ϕ (here $d\colon \mathcal{O}_X \to \Omega_X$ is the canonical differential), and $\mathcal{F}$ an $\mathcal{O}_X$-module. In the first chapter, we construct a sequence of sheaves $\mathcal{P}^r_\phi(\mathcal{F})$ over X, indexed by integers $r \geq 0$, and depending on M, ϕ, $\mathcal{F}$, together with, for each $r \geq 0$, a differential operator $\mathcal{F} \to \mathcal{P}^r_\phi(\mathcal{F})$ of order $\leq r$.

Analogously to the construction of $\mathcal{P}^r(\mathcal{F})$ described in the previous section, the construction of $\mathcal{P}^r_\phi(\mathcal{F})$ begins with the special case $\mathcal{P}^r_\phi = \mathcal{P}^r_\phi(\mathcal{O}_X)$; $\mathcal{P}^r_\phi$ is a sheaf of rings, and comes equipped with two morphisms $\psi^r_{\mathrm{I}}, \psi^r_{\mathrm{II}}\colon \mathcal{O}_X \to \mathcal{P}^r_\phi$ of sheaves of rings (again analogously to the $\mathcal{P}^r$ case), which therefore induce two $\mathcal{O}_X$-module structures on $\mathcal{P}^r_\phi$.

We adopt the convention that we work with the ψ_{I}-induced $\mathcal{O}_X$-module structure on $\mathcal{P}^r_\phi$. In this case, ψ^r_{II} is shown in Chapter 2 to be a differential operator of order $\leq r$.

Then, we define

$$\mathcal{P}^r_\phi(\mathcal{F}) = \mathcal{P}^r_\phi \,{}_{\mathrm{II}}\!\otimes \mathcal{F},$$

where the symbol ${}_{\mathrm{II}}\otimes$ indicates that the $\mathcal{P}^r_\phi$ on the left-side of the tensor product is taken with its ψ^r_{II}-induced $\mathcal{O}_X$-module structure. The differential operator $\mathcal{F} \to \mathcal{P}^r_\phi(\mathcal{F})$ is defined to be the sheaf map $f \mapsto 1_{\mathcal{P}^r_\phi} \otimes f$, where $1_{\mathcal{P}^r_\phi}$ is the multiplicative identity for the ring structure of $\mathcal{P}^r_\phi$, and f is a local section of $\mathcal{F}$.

When $\mathcal{F} = L$ is a line bundle, the sheaf $\mathcal{P}^r_\phi(L)$ is locally free of rank $r+1$ (one good feature of $\mathcal{P}^r_\phi(L)$ is that this is true even when X is singular!). We now give a local description of the differential operator $L \to \mathcal{P}^r_\phi(L)$, which may illuminate how the bundle $\mathcal{P}^r_\phi(L)$ depends on the choice of morphism $\phi\colon \Omega_X \to M$.

Suppose that $s\colon U \to M$ and $t\colon U \to L$ are local frames of M and L, respectively, over an open set $U \subset X$, let $\xi = f\,t$, $f \in \mathcal{O}_X(U)$ be a section of L, and define the symbol $D^k_{s,t}(\xi)$

inductively by:

$$D^0_{s,t}(\xi) = f, \quad D(f) =: D^1_{s,t}(\xi)\, s, \quad \ldots, \quad D(D^k_{s,t}(\xi)) =: D^{k+1}_{s,t}(\xi)\, s, \quad \ldots.$$

Then there is an induced frame for $\mathcal{P}^r_\phi(L)$ over U (that depends on s and t) with respect to which the morphism $L \to \mathcal{P}^r_\phi(L)$ takes the form

$$f \mapsto \left(\frac{D^0_{s,t}(\xi)}{0!}, \frac{D^1_{s,t}(\xi)}{1!}, \frac{D^2_{s,t}(\xi)}{2!}, \ldots, \frac{D^k_{s,t}(\xi)}{k!}, \ldots, \frac{D^r_{s,t}(\xi)}{r!} \right), \tag{1.2}$$

which we may informally think of as a Taylor expansion of ξ in the derivation $D = \phi \circ d$.

When X is smooth and ϕ is surjective, we can interpret the derivation D associated to ϕ as a directional derivative along directions prescribed by ϕ. This proceeds as follows. The kernel of ϕ is a rank $(\dim X - 1)$ subbundle of Ω_X, and we have a short exact sequence

$$0 \to \ker(\phi) \to \Omega_X \to M \to 0,$$

which dualizes to the short exact sequence

$$0 \to M^\vee \to T_X \to \ker(\phi)^\vee \to 0.$$

Therefore, in this case, the data of ϕ is equivalent to a foliation $M^\vee \subset T_X$.

For each point p of X, the derivation $D = \phi \circ d$ associated to ϕ is the directional derivative in the direction $(M^\vee)_p$. Therefore, we may roughly think of $\mathcal{P}^r_\phi(L)$ as a vector bundle that records the first r iterations of a directional derivative of sections of L along directions that are prescribed by the choice of morphism ϕ.

A good feature of the bundles $\mathcal{P}^r_\phi(L)$ is that, over any X (including the singular case), the $\mathcal{P}^r_\phi(L)$ fit into the short exact sequence

$$0 \to M^{\otimes r} \otimes L \to \mathcal{P}^r_\phi(L) \to \mathcal{P}^{r-1}_\phi(L) \to 0$$

that is analogous to the sequence (1.1). In particular, combining this short exact sequence with induction on r, we show in Chapter 2 that the Chern classes of $\mathcal{P}^r_\phi(L)$ are given by

$$c_m(\mathcal{P}^r_\phi(L)) = \sum_{k=0}^{m} \sigma(r+1, r+1-k) \binom{r+1-k}{m-k} c_1(L)^{m-k} c_1(M)^k,$$

where $\sigma(n,k)$ are Stirling numbers of the first kind.

Another good feature of the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$ is that they behave well with respect to pullback. More precisely, let $\pi\colon Y \to X$ be a morphism of schemes, M be a line bundle on X, $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $\phi_X\colon \Omega_X \to M$ and $\phi_Y\colon \Omega_Y \to \pi^*M$ be morphisms of $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules, respectively, such that $\pi^*\phi_X = \phi_Y \circ d\pi$. For each integer $r \geq 0$, there exists a morphism

$$\gamma^r(\mathcal{F})\colon \pi^*\left(\mathcal{P}^r_{\phi_X}(\mathcal{F})\right) \to \mathcal{P}^r_{\phi_Y}(\pi^*\mathcal{F})$$

that is an isomorphism of $\mathcal{O}_Y$-algebras (the pullback of $\mathcal{P}^r_{\phi_X}(\mathcal{F})$ is taken with respect to its ψ^r_{I}-induced $\mathcal{O}_X$-module structure, and $\mathcal{P}^r_{\phi_Y}(\pi^*\mathcal{F})$ is taken with its ψ_{I}-induced $\mathcal{O}_Y$-module structure). The isomorphism $\gamma^r(\mathcal{F})$ is also well-behaved with respect to global sections.

The above behaviour of $\mathcal{P}^r_\phi(\mathcal{F})$ under pullback is crucial for both of the applications of $\mathcal{P}^r_\phi(L)$ studied in Chapters 3 and 4.

Chapter 3. Weierstrass weight absorbed by a singularity. The remaining two chapters of the thesis are devoted to applications of the sheaves $\mathcal{P}^r_\phi(\mathcal{F})$.

Some of the motivation for the investigations of Chapter 3 arises from our attempt to generalize the following two classical examples — in a one-parameter family of elliptic curves that degenerate to a cuspidal cubic, exactly eight of the nine flex points are absorbed into the cusp; similarly, in a one-parameter family of elliptic curves that degenerate to a nodal cubic, exactly six flex points are absorbed into the node.

In Chapter 3, using $\mathcal{P}^r_\phi(L)$, we extend this pair of results to situations in which a collection of Weierstrass points (of a family of linear series $\{|V_t|\}_{t\in\mathbb{P}^1}$ on a family of curves $\{C_t\}_{t\in\mathbb{P}^1}$, where each C_t is contained in a fixed smooth surface) approaches a singularity.

To compute how many Weierstrass points (counted with multiplicity equal to their Weierstrass weight) get absorbed into the singular point, we proceed as follows.

Let C be a curve contained in a smooth surface S, $p \in C$ be a singular point, L be a line bundle on C, and $V \subset H^0(C, L)$ be a linear series. We define the Weierstrass weight of $|V| = \mathbb{P}V$ at p by analogy with the smooth case, namely as the order of vanishing of the determinant of the morphism

$$V \otimes_k \mathcal{O}_C \to \mathcal{P}^r_\phi(L),$$

where the morphism $\phi \colon \Omega_C \to \omega_C$ is constructed by choosing a local equation $f = 0$ of C over an open set $U \subset S$, restricting the local morphisms $\Omega_S(U) \to K_S(C)(U)$ given by

$$\omega \mapsto \omega \wedge \frac{df}{f}$$

to C, and checking that the restrictions do not depend on the choice of f, which allows gluing the restrictions to a global morphism ϕ.

We prove in Chapter 3 that the construction of ϕ is well-behaved in families (in a precise sense), and, as a consequence, that in a neighbourhood of the singular fibre, the sum of the Weierstrass weights of the points that are absorbed into the singular point is equal to the Weierstrass weight at the singular point.

We also develop a way of computing the Weierstrass weight of a (possibly singular) point in terms of quantities defined on the normalization of C. This proceeds as follows.

Let $\pi \colon \tilde{C} \to C$ be the normalization of C, Δ be the adjoint divisor of π (a clasically-studied divisor on $\tilde{C}$), and $\cdot\Delta \colon \Omega_{\tilde{C}} \to \Omega_{\tilde{C}}(\Delta)$ be the morphism given by multiplication of the local equation of Δ. We show that

$$\pi^* \left(\mathcal{P}^r_\phi(L)\right) \cong \mathcal{P}^r_{\cdot\Delta}(\pi^* L),$$

and deduce that the Weierstrass weight at a (possibly singular) point $p \in C$ is equal to

$$\binom{n+1}{2} \deg \Delta + \sum_{r \in \pi^{-1}(p)} \tilde{w}(r),$$

where $n+1 = \dim_k V$, and $\tilde{w}(r)$ denotes the Weierstrass weight of the pulled-back linear series $\pi^* V$. We find this result to be of independent interest, but it also yields a convenient way of computing the Weierstrass weight at a singular point and, therefore, of computing how much Weierstrass weight is absorbed into a singular point in a curve degeneration.

Chapter 4. Counting lines in a linear system in a Grassmannian. In Chapter 4, using $\mathcal{P}^r_\phi(L)$, we count the number of lines that appear in a linear system in a Grassmannian (in its Plücker embedding).

Let $F_1(X)$ denote the Fano variety of lines in $X \subset \mathbb{P}^N$, $\mathcal{U}_1(X)$ denote the universal family over $F_1(X)$, L be a line bundle on X, and $V \subset H^0(X, L)$ be a linear series (of a degree d and dimension $n+1$, where d and $n+1$ are chosen such that finitely many lines are expected to appear in $|V|$). To set up the count of the number of lines that appear in $|V|$ as a Chern class computation, we consider the projections

$$\begin{tikzcd} \mathcal{U}_1(X) \arrow[r, "\mathbb{P}^1\text{-bundle}", "\pi_F"'] \arrow[d, "\pi_X"] & F_1(X) \\ X. & \end{tikzcd}$$

The morphism ϕ (that is used for $\mathcal{P}^r_\phi(L)$ in Chapter 4) is by definition the morphism given by the cotangent sheaf exact sequence

$$\pi_F^* \Omega_{F_1(X)} \to \Omega_{\mathcal{U}_1(X)} \xrightarrow{\phi} \Omega_{\mathcal{U}_1(X)/F_1(X)} \to 0.$$

We show that the (rational equivalence class of the) locus of lines that appear in $|V|$ is (the pushforward by π_X of) the locus of points where the morphism

$$V \otimes_k \mathcal{O}_{\mathcal{U}_1(X)} \to \mathcal{P}^r_\phi(\pi_X^* L)$$

does not have full rank, which by the Thom-Porteous formula is equal to

$$\pi_{X,*}\ c_{\dim F_1(X)}(\mathcal{P}^d_\phi(\pi_X^* L)).$$

For $X = \mathbb{G}(k,n)$ in its Plücker embedding, we obtain the following formula: setting

$$B_i := \begin{cases} \binom{n-1}{k}, & i = n-1, \\ \sum_{a=m_i}^{M_i} \binom{i+1}{a} \frac{(i+1-a)!}{(i-n-a)!} \frac{1}{(n-k-1-a)!\,(n+k-i+a)!}, & i > n-1, \end{cases}$$

where

$$m_i = \max(i-n-k,\, 0),$$
$$M_i = \min(i-n,\, n-k-1),$$

the number of lines that appear in a linear system of degree d and projective dimension $d-(k+1)(n-k)-n+3$ (and subject to certain additional hypotheses) is

$$\prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!} \sum_{i=n-1}^{2n-1} (-1)^{i-n+1} \sigma(d+1,\, d+1-i) \binom{d+1-i}{f-i} B_i\, d^{f-i} \frac{(f-i+1)!}{i-n+1},$$

where $\sigma(n,k)$ denote Stirling numbers of the first kind.

As special cases, we recover the well-known facts that a cubic surface in $\mathbb{P}^3$ is expected to contain 27 lines, a pencil of K3 surfaces in $\mathbb{P}^3$ 320 lines, a quintic threefold in $\mathbb{P}^4$ 2875 lines and, perhaps less well-known, a degree 4 hypersurface in $\mathbb{G}(1,3)$ 1280 lines.

1.3 Intended Application to Newton–Okounkov Bodies

The purpose of this section is to describe the connection of the origins of this work to a branch of current research, namely the theory of Newton–Okounkov bodies.

The theory of Newton–Okounkov bodies is a relatively recent development within Algebraic Geometry, with the two founding papers of the field ([LM09] and [KK12]) appearing in publication in 2009 and 2012.

One of the origins of the theory of Newton–Okounkov bodes lies in the fact that the volume of a line bundle L on an irreducible variety X of dimension n, which is defined as

$$\mathrm{vol}(L) = \mathrm{vol}_X(L) = \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!}$$

([Laz04, § 2.2.C]) has the log-concavity property

$$\mathrm{vol}(L \otimes L')^{1/n} \geq \mathrm{vol}(L)^{1/n} + \mathrm{vol}(L')^{1/n}, \tag{1.3}$$

where L, L' are any two big line bundles ([Laz04, Remark 2.2.50]). In [Oko03], Okounkov was able to define, given a line bundle L, a convex body in $\mathbb{R}^n$ whose Euclidean volume was equal to $\mathrm{vol}_X(L)$. The log-concavity property (1.3) was then obtained as an application of the Brunn–Minkowski inequality. The aforementioned papers of Lazarsfeld–Mustata and Kaveh–Khovanskii then systematically studied Okounkov's construction, and laid the foundations of this subfield.

We now define a Newton–Okounkov body.

Let X be an irreducible variety of dimension n over an algebraically closed field k, and let

$$Y_\bullet \quad : \quad Y_0 \supset Y_1 \supset \cdots \supset Y_{n-1} \supset Y_n$$

be a flag of irreducible subvarieties of X, such that $\operatorname{codim}_X Y_i = i$ for all $0 \leq i \leq n$, and Y_{i+1} is a Cartier divisor in Y_i for each $0 \leq i \leq n-1$. For any line bundle M on X, a function

$v_{Y_\bullet} : H^0(X,M) \smallsetminus \{0\} \to \mathbb{Z}^n$ is defined as follows. Let $s \in H^0(X,M) \smallsetminus \{0\}$.

First, define

$$v_1(s) = \mathrm{ord}_{Y_1}(s).$$

Let f_1 be the local equation of Y_1 in Y_0 in a neighbourhood of Y_n. Then, dividing s by $f_1^{v_1(s)}$, we obtain a section $\tilde{s_1}$ of $H^0(Y_0, L \otimes \mathcal{O}_{Y_0}(-v_1(s)Y_1))$ that does not identically vanish along Y_1. Let $s_1 = \tilde{s_1}|_{Y_1}$, set

$$v_2(s) = \mathrm{ord}_{Y_2}(s_1)$$

and proceed similarly to define $v_3(s), \dots, v_n(s)$.

Now, let L be a big line bundle over X. The Newton–Okounov body $\Delta_{Y_\bullet}(L)$ associated to L and $Y_\bullet$ is defined as follows.

$$\Delta_{Y_\bullet}(L) = \text{Closed convex hull}\left(\bigcup_{m=1}^{\infty} v_{Y_\bullet}\left(H^0(X, L^{\otimes m}) \smallsetminus \{0\}\right)\right).$$

We say that a Newton–Okounkov body is finitely-generated if there exists an integer $N \geq 1$ so that

$$\Delta_{Y_\bullet}(L) = \text{Closed convex hull}\left(\bigcup_{m=1}^{N} v_{Y_\bullet}\left(H^0(X, L^{\otimes m}) \smallsetminus \{0\}\right)\right).$$

Note that if $\Delta_{Y_\bullet}(L)$ is finitely-generated, then it is a convex polytope.

The paper [And13] of Dave Anderson shows that if $\Delta_{Y_\bullet}(L)$ is finitely-generated, then there exists a degeneration of X to a variety whose normalization is the toric variety that corresponds to $\Delta_{Y_\bullet}(L)$.

This interesting result (that was applied to the construction of integrable systems by Harada and Kaveh in [HK15]) raises the problem of finding criteria for finite-generation of Newton–Okounkov bodies.

In unpublished work, we have studied how $v_{Y_\bullet}(H^0(X,L) \smallsetminus \{0\})$ varies as $Y_\bullet$ varies in a flat family over an irreducible base T. To describe the result, let

$$\mathcal{Y}_\bullet \quad : \quad \mathcal{Y}_0 \supset \cdots \supset \mathcal{Y}_n,$$

be a flag, with $\mathcal{Y}_0 = X \times T$, with each $\mathcal{Y}_i$ flat and surjective over T, and with $Y_{i,t}$ an irreducible Cartier divisor in $Y_{i-1,t}$ for each $t \in T$ and each $1 \le i \le n$. Let L be a line bundle over X. Then one can show that the collection of images $v_{Y_\bullet,t}(H^0(X,L) \smallsetminus \{0\})$ as t varies over *all* points of T is finite, and therefore can be totally ordered! (This fact was shown in the paper of Lazarsfeld–Mustata [LM09].)

We have been able to prove that T is stratified by the image of the valuation $v_{Y_\bullet}$, in the following sense:

Write

$$\bigcup_{t \in T} v_{Y_\bullet,t}\left(H^0(X,L) \smallsetminus \{0\}\right) = \{\omega_0, \ldots, \omega_R\},$$

with $\omega_i \le \omega_{i+1}$ in lexicographic order on $\mathbb{Z}^d$. Suppose that $h^0(X,L) = r+1$. For a Young diagram $\lambda = (\lambda_0, \ldots, \lambda_r)$ fitting inside a box with $r+1$ rows and $R-r$ columns, define

$$W_\lambda := \left\{t \in T \,:\, v_{Y_\bullet,t}\left(H^0(X,L) \smallsetminus \{0\}\right) = \left\{\omega_{0+\lambda_r}, \omega_{1+\lambda_{r-1}}, \ldots, \omega_{r+\lambda_0}\right\}\right\} \subset T.$$

In unpublished work, we have been able to prove the following theorem.

Theorem. *The subsets W_λ make up a locally-closed stratification of T. If the subset W_λ is contained in the closure of W_μ , then the diagram μ dominates λ.*

Examples. (i) When X is a curve, it is natural to choose $T = X$ and $\mathcal{Y}_\bullet : X \times X \supset \Delta_{X \times X}$. The subsets W_λ are: the open set of non-Weierstrass points, and the sets of Weierstrass points with some fixed vanishing sequence.

(ii) Let $X \subset \mathbb{P}^3$ be a smooth cubic surface and $L = \mathcal{O}_{\mathbb{P}^3}(2)|_X$. Consider the family of flags composed of a fixed line $\ell \subset X$ and a varying point p of ℓ. Restricting L to ℓ gives a 2:1 map from $\mathbb{P}^1$ to itself. The image of $\nu_{Y_\bullet}$ depends on whether or not p is a ramification point of this map:

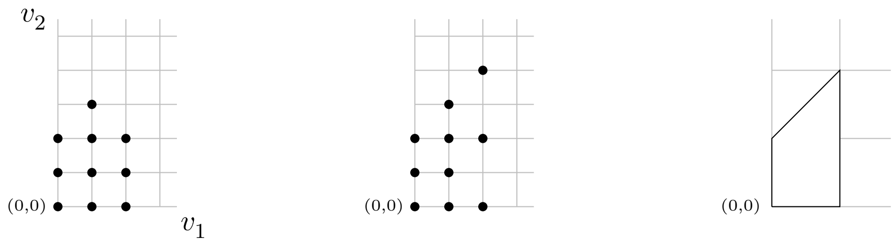


Figure 1.2: On the left: $v_{X\supset\ell\supset p}(H^0(X,L)\smallsetminus\{0\})$ when p is not a ramification point (such flags define the locus W_0 in T). In the center: $v_{X\supset\ell\supset p}(H^0(X,L)\smallsetminus\{0\})$ when p is a ramification point (such flags define the locus $W_{(1,1)}$ in T). On the right: the Okounkov body of L, which does not depend on the flag as a convex body, but which is finitely-generated when p is a ramification point and not finitely-generated when p is not a ramification point.

The theorem suggests that if $\Delta_{Y_\bullet,t}(L)$ is not finitely generated for some $t\in T$, then it is not finitely-generated for all t in a very general subset of T (but we do not have a proof of this in general).

Because general families of flags whose base variety has sufficiently high dimension contain flags whose Y_n is a singular point of at least one of the Y_i, to better understand the problem of varying Newton–Okounkov bodies in families, it appears necessary to extend the definition of $\Delta_{Y_\bullet}(L)$ to such flags.

The investigation of the behaviour of Weierstrass weight in curve degenerations that appears in Chapter 3 of the present thesis is a first step towards studying Okounkov bodies where Y_n is a singular point of some Y_i.

Although the results of this thesis have no direct relation with the theory of Newton–Okounkov bodies, our attempts at studying the variation of Newton–Okounkov bodies with the flag led us to investigate the modification of the usual principal parts sheaves that is presented here. In future work, we intend to apply $\mathcal{P}^r_\phi(\mathcal{F})$ to further investigate the variation of Newton–Okounkov bodies.

1.4 Conventions

(Unless noted otherwise:)

The symbol $\mathbb{N}$ denotes the nonnegative integers.

The symbol $\subset$ denotes arbitrary inclusion (not necessarily proper); the symbol $\subsetneq$ denotes proper inclusion.

Sheaves are denoted by calligraphic roman letters, with the exception that vector bundles and their associated locally free sheaves of sections are identified with each other, and are usually both denoted by capital roman letters.

Schemes are assumed to be reduced and finite-type over an algebraically closed field k of characteristic zero.

Notation for derivations — let X be a scheme, M be a line bundle over X, and $\phi \colon \Omega_X \to M$ be a morphism of $\mathcal{O}_X$-modules. Precomposing ϕ with the differential $d \colon \mathcal{O}_X \to \Omega_X$ gives a derivation $\phi \circ d \colon \mathcal{O}_X \to M$, which (unless otherwise noted) will be denoted by D_ϕ (or simply D) in what follows. Conversely, by the universal property of Ω_X, a derivation $D \colon \mathcal{O}_X \to M$ induces a map $\phi_D \colon \Omega_X \to M$ of $\mathcal{O}_X$-modules such that $D = \phi_D \circ d$.

Chapter 2

Construction and basic properties of sheaves of ϕ-principal parts $\mathcal{P}_\phi^r(\mathcal{F})$

Let X be a scheme, M a line bundle over X, and $\phi\colon \Omega_X \to M$ a morphism of $\mathcal{O}_X$-modules.

We begin this chapter with an inductive construction of the collection $\{\mathcal{P}_\phi^r, \psi_{\mathrm{I}}^r, \psi_{\mathrm{II}}^r, \pi^r\}_{r\in\mathbb{N}}$, where $\mathcal{P}_\phi^r$ are sheaves of rings over X, depending on ϕ, equipped with maps $\psi_{\mathrm{I}}^r, \psi_{\mathrm{II}}^r\colon \mathcal{O}_X \to \mathcal{P}_\phi^r$ of sheaves of rings (ψ_{I}^r and ψ_{II}^r therefore induce two $\mathcal{O}_X$-module structures on $\mathcal{P}_\phi^r$), and $\pi^r\colon \mathcal{P}_\phi^r \to \mathcal{P}_\phi^{r-1}$ are projection morphisms.

The analogy between $\mathcal{P}_\phi^r$ and the usual sheaves of principal parts $\mathcal{P}^r$ was introduced in Chapter 1; the interpretation of the morphism $\psi_{\mathrm{II}}^r\colon \mathcal{O}_X \to \mathcal{P}_\phi^r$ as a kind of r-th order Taylor expansion in the derivation $D = \phi\circ d$ (when $\mathcal{P}_\phi^r$ is taken with its ψ_{I}^r-induced $\mathcal{O}_X$-module structure) was likewise mentioned in the introduction.

Our interest in constructing and studying the sheaves $\mathcal{P}_\phi^r$ arises mainly from their applications, two of which are described in Chapters 3 and 4. In Chapter 2, we will focus on establishing formal properties of $\mathcal{P}_\phi^r$, many of which will be useful for studying the two applications.

We now summarize the results that are established in this chapter.

Following the inductive construction of $\{\mathcal{P}_\phi^r, \psi_{\mathrm{I}}^r, \psi_{\mathrm{II}}^r, \pi^r\}_{r\in\mathbb{N}}$, we show (in Corollary 2.3) that the projections π^r fit into the short exact sequence

$$0 \longrightarrow M^{\otimes r} \longrightarrow \mathcal{P}^r_\phi \xrightarrow{\pi^r} \mathcal{P}^{r-1}_\phi \longrightarrow 0$$

and conclude by induction on r that the $\mathcal{P}^r_\phi$ are always locally free (with respect to either of the two $\mathcal{O}_X$-module structures), even when X has singularities.

Because the sheaves $\mathcal{P}^r_\phi$ are always locally free, they can be useful for extending constructions made using the usual principal parts sheaves to situations when X is singular — an application along these lines is described in Chapter 3.

For fixed r and $1 \le k \le r$, we define the vector subbundle $\mathcal{N}_{r,k} \subset \mathcal{P}^r_\phi$ as the kernel of the composition $\pi^k \circ \cdots \circ \pi^{r-1} \circ \pi^r : \mathcal{P}^r_\phi \to \mathcal{P}^{k-1}_\phi$, and show (in Theorem 2.7) that the sequence $\{\mathcal{N}_{r,k}\}_k$ defines a multiplicative decreasing filtration of $\mathcal{P}^r_\phi$.

The filtration by $\mathcal{N}_{r,k}$ is a basic property of $\mathcal{P}^r$ in its own right, but also provides a tool for proving (in Corollary 2.9) that the morphism ψ_{II} is a differential operator of order $\le r$ with respect to the ψ_{I}-induced $\mathcal{O}_X$-module structure on $\mathcal{P}^r$ (and that reciprocally, ψ_{I} is a differential operator of order $\le r$ with respect to the ψ_{II}-induced $\mathcal{O}_X$-module structure).

For an $\mathcal{O}_X$-module $\mathcal{F}$, we define $\mathcal{P}^r_\phi$-modules

$$\mathcal{P}^r_\phi(\mathcal{F}) = \mathcal{P}^r_{\phi\ \mathrm{II}}\!\otimes \mathcal{F}$$

and extend to $\mathcal{P}^r_\phi(\mathcal{F})$ both the differential operator ψ_{II}, and the filtration of $\mathcal{P}^r_\phi$ (the extensions are denoted $\psi_{\mathrm{II}}(\mathcal{F})$ and $\mathcal{N}_{r,k}(\mathcal{F})$, respectively). At the end of the section, for $V \subset H^0(X, \mathcal{F})$, we construct the morphism

$$V \otimes_k \mathcal{O}_X \to \mathcal{P}^r_\phi(\mathcal{F}),$$

which will be very useful in Chapters 3 and 4.

In §2.3, we describe in local coordinates the product structure of $\mathcal{P}^r$, the morphisms $\psi_{\mathrm{I}}, \psi_{\mathrm{II}}, \pi^r$, and the filtration by $\mathcal{N}_{r,k}$. This description is useful technically for performing local computations, and is intended to illuminate the inductive construction of $\mathcal{P}^r_\phi$. In subsection 1.3.1, we describe a method for computing the transition functions of the vector bundles $\mathcal{P}^r_\phi$, which is not directly applied elsewhere in the thesis, but which we find to be of independent interest.

The lengthy §2.4 culminates in Theorem 2.24, which we will refer to as the functoriality theorem, and which will be key for both of the applications of Chapters 3 and 4. The statement is:

Theorem 2.24. *Let $\pi : Y \to X$ be a morphism of schemes, M be a line bundle on X, $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $\phi_X : \Omega_X \to M$ and $\phi_Y : \Omega_Y \to \pi^* M$ be derivations such that $\pi^*\phi_X = \phi_Y \circ d\pi$. For each integer $r \geq 0$, there exists a morphism*

$$\gamma^r(\mathcal{F}) : \pi^*\left(\mathcal{P}^r_{\phi_X}(\mathcal{F})\right) \to \mathcal{P}^r_{\phi_Y}(\pi^*\mathcal{F})$$

that is an isomorphism of $\mathcal{O}_Y$-algebras (the pullback of $\mathcal{P}^r_{\phi_X}(\mathcal{F})$ is taken with respect to its ψ^r_{I}-induced $\mathcal{O}_X$-module structure, and $\mathcal{P}^r_{\phi_Y}(\pi^*\mathcal{F})$ is taken with its ψ_{I}-induced $\mathcal{O}_Y$-algebra structure).

Moreover, let $V \subset H^0(X, \mathcal{F})$, and $\pi^* V$ be the pulled-back linear series. Then the diagram

$$\begin{array}{ccc} \pi^*(V \otimes_k \mathcal{O}_X) & \longrightarrow & \pi^*_{\mathrm{I}}\left(\mathcal{P}^r_{\phi_X}(\mathcal{F})\right) \\ \downarrow & & {\scriptstyle \gamma^r_{\mathrm{I}}(\mathcal{F})}\downarrow{\scriptstyle \cong} \\ \pi^* V \otimes_k \mathcal{O}_Y & \longrightarrow & \mathcal{P}^r_{\phi_Y}(\pi^*\mathcal{F}) \end{array}$$

commutes.

We kindly refer the reader to the introduction of section §2.4 for a more detailed outline of the proof of Theorem 2.24.

In §2.5, we show as a consequence of the functoriality theorem that the bundles $\mathcal{P}^r_\phi(E)$ (where E is a vector bundle) are well-behaved in families. This result will be applied in Chapter 3.

We finish the chapter by finding in §2.6 general formulas for the Chern classes $c_m(\mathcal{P}^r_\phi(L))$ in terms of the Chern classes of L and M. Namely, we show that (Theorem 2.29) for any $0 \le m \le r+1$, we have

$$c_m(\mathcal{P}^r_\phi(L)) = \sum_{k=0}^{m} \sigma(r+1, r+1-k) \binom{r+1-k}{m-k} c_1(L)^{m-k} c_1(M)^k,$$

where $\sigma(n,k)$ denote Stirling numbers of the first kind. These formulas will be especially useful in Chapter 4.

2.1 Inductive construction of $\mathcal{P}^r_\phi$

Our construction of $\{\mathcal{P}^r_\phi, \psi^r_{\mathrm{I}}, \psi^r_{\mathrm{II}}, \pi^r\}_{r\in\mathbb{N}}$ proceeds by induction on r:

r = 0: Set

$$\mathcal{P}^0_\phi := \mathcal{O}_X \qquad \text{and} \qquad \psi^0_{\mathrm{I}} = \psi^0_{\mathrm{II}} = \mathrm{id}_{\mathcal{O}_X}.$$

The ring structure on $\mathcal{P}^0_\phi$ defined to be the usual ring structure on $\mathcal{O}_X$. It is clear that ψ^0_{I} and ψ^0_{II} are maps of sheaves of rings; consequently, ψ^0_{I} and ψ^0_{II} induce two $\mathcal{O}_X$-module structures on $\mathcal{P}^0_\phi$ in the usual way.[1] (The two $\mathcal{O}_X$-module structures coincide for $\mathcal{P}^0_\phi$ for the trivial reason that $\psi^0_{\mathrm{I}} = \psi^0_{\mathrm{II}}$, but do not in general coincide for $r > 0$.)

r = 1: Set

$$\mathcal{P}^1_\phi := \mathcal{O}_X \oplus M \qquad \text{and} \qquad \begin{aligned} \psi^1_{\mathrm{I}} &:= \mathrm{id}_{\mathcal{O}_X} \oplus 0, \\ \psi^1_{\mathrm{II}} &:= \mathrm{id}_{\mathcal{O}_X} \oplus D. \end{aligned}$$

(Here, as above, $D = \phi \circ d$ is the derivation induced by ϕ.)

[1] For an open subset $U \subset X$, $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{P}^0_\phi(U) = \mathcal{O}_X(U)$, the ψ_{I}-induced $\mathcal{O}_X(U)$-module structure on $\mathcal{P}^0_\phi(U)$ is defined by $f \cdot s := \psi^0_{\mathrm{I}}(f)\, s$, where the product on the right-hand side of the equality is taken in the ring $\mathcal{P}^0_\phi(U)$; similarly, the ψ_{II}-induced $\mathcal{O}_X(U)$-module structure on $\mathcal{P}^0_\phi(U)$ is defined by $f \cdot s := \psi^0_{\mathrm{II}}(f)\, s$.

The map π^1 is defined to be projection onto the first direct summand:

$$\begin{tikzcd} \mathcal{O}_X \oplus M \arrow[r] \arrow[d, equal] & \mathcal{O}_X \arrow[d, equal] \\ \mathcal{P}^1_\phi \arrow[r, dashed, "\pi^1", "\text{def.}"'] & \mathcal{P}^0_\phi \end{tikzcd}$$

The ring structure on $\mathcal{P}^1_\phi$ is defined as follows. Let $U \subset X$ be open, and let (f, s) and (g, t) be sections of $\mathcal{P}^1_\phi(U) = \mathcal{O}_X(U) \oplus M(U)$. Define

$$(f, s) \cdot (g, t) := (fg,\ ft + gs).$$

The section $(1, 0)$ is the multiplicative identity, and is denoted by $1_{\mathcal{P}^1_\phi}$ or simply 1.

The maps ψ^1_{I} and ψ^1_{II} are morphisms of sheaves of rings,[2] hence ψ^1_{I} and ψ^1_{II} induce two $\mathcal{O}_X$-module structures on $\mathcal{P}^1_\phi$.

r ≥ 2: Set

$$\mathcal{P}^r_\phi := \mathcal{O}_X \oplus \left(\mathcal{P}^{r-1}_\phi \ {}_{\mathrm{II}}\!\otimes M\right) \qquad \text{and} \qquad \begin{aligned} \psi^r_{\mathrm{I}} &:= \mathrm{id}_{\mathcal{O}_X} \oplus 0, \\ \psi^r_{\mathrm{II}} &:= \mathrm{id}_{\mathcal{O}_X} \oplus \left(1_{\mathcal{P}^{r-1}_\phi} \otimes D\right). \end{aligned} \tag{2.1}$$

The symbol ${}_{\mathrm{II}}\!\otimes$ indicates that the tensor product is taken with respect to the $\mathcal{O}_X$-module structure on $\mathcal{P}^{r-1}_\phi$ that is induced by ψ^{r-1}_{II}; that is, for $U \subset X$ open, and sections f of $\mathcal{O}_X$ and $g \otimes s$ of $\mathcal{P}^{r-1}_\phi \ {}_{\mathrm{II}}\!\otimes M$ over U,

$$f(g \otimes s) = (\psi^{r-1}_{\mathrm{II}}(f) g) \otimes s = g \otimes (fs).$$

[2]For ψ^1_{II}, this is a consequence of the fact that D is a derivation— for an open set $U \subset X$ and a section f of $\mathcal{O}_X$ over U, $\psi^1_{\mathrm{II}}(fg) = (fg,\ D(fg)) = (fg,\ fDg + gDf) = (f,\ Df) \cdot (g,\ Dg) = \psi^1_{\mathrm{II}}(f) \cdot \psi^1_{\mathrm{II}}(g)$.

The map π^r is constructed inductively:

$$\begin{tikzcd} \mathcal{O}_X \oplus \left(\mathcal{P}^{r-1}_\phi \ {}_{\mathrm{II}}\!\otimes M\right) \arrow[r, "\mathrm{id}_{\mathcal{O}_X} \oplus (\pi^{r-1} \otimes \mathrm{id}_M)"] \arrow[d, equal] & \mathcal{O}_X \oplus \left(\mathcal{P}^{r-2}_\phi \ {}_{\mathrm{II}}\!\otimes M\right) \arrow[d, equal] \\ \mathcal{P}^r_\phi \arrow[r, dashed, "\pi^r", "\text{def.}"'] & \mathcal{P}^{r-1}_\phi \end{tikzcd}$$

The sheaf $\mathcal{P}^{r-1}_\phi \ {}_{\mathrm{II}}\!\otimes M$ is a $\mathcal{P}^{r-1}_\phi$-module in the natural way, which lets us define a product structure on $\mathcal{P}^r_\phi$ as follows. Let $U \subset X$ be open, and let (f,s) and (g,t) be sections of $\mathcal{P}^r_\phi(U) = \mathcal{O}_X(U) \oplus \left(\mathcal{P}^{r-1}_\phi \ {}_{\mathrm{II}}\!\otimes M\right)(U)$. Define

$$(f,s) \cdot_{\mathcal{P}^r_\phi(U)} (g,t) = (fg,\ \pi^r(f,s)t + \pi^r(g,t)s).$$

The multiplicative identity is equal to the section $1 \oplus 0$, which we denote by $1_{\mathcal{P}^r_\phi}$ or simply 1.

The maps ψ^r_{I} and ψ^r_{II} are morphisms of sheaves of rings, hence induce two $\mathcal{O}_X$-module structures on $\mathcal{P}^r_\phi$.

Remark. The inductive step makes sense with $r = 1$, and in this case produces the same $\mathcal{P}^1_\phi$ (including the ring structure), ψ^1_{I}, ψ^1_{II} and π^1 as constructed above, so it is sufficient to take $r = 0$ as the base case.

2.2 Basic properties of $\mathcal{P}^r_\phi(\mathcal{F})$

Throughout this section (unless otherwise noted), r denotes an integer ≥ 1.

Proposition 2.1. *The map π^r is a morphism of sheaves of rings. We have $\psi^{r-1}_{\mathrm{I}} = \pi^r \circ \psi^r_{\mathrm{I}}$ and $\psi^{r-1}_{\mathrm{II}} = \pi^r \circ \psi^r_{\mathrm{II}}$. Consequently, π^r is a morphism of $\mathcal{O}_X$-algebras with respect to either the $\mathcal{O}_X$-module structures on $\mathcal{P}^r_\phi$ and $\mathcal{P}^{r-1}_\phi$ induced by the maps ψ^r_{I} and ψ^{r-1}_{I} (resp.), or the $\mathcal{O}_X$-module structures induced by the maps ψ^r_{II} and ψ^{r-1}_{II} (resp.).*

Proof. The first two statements can be verified by simple computations (and induction on r). The last statement follows immediately by combining the first two statements. $\square$

Proposition 2.2. *The kernel of* π^r *is isomorphic to* $M^{\otimes r}$ *(as an* $\mathcal{O}_X$*-module) with respect to either the* ψ_{I}^r *or* ψ_{II}^r $\mathcal{O}_X$*-module structure on* $\mathcal{P}_\phi^r$*. Consequently, the* ψ_{I}^r *and* ψ_{II}^r $\mathcal{O}_X$*-module structures of* $\mathcal{P}_\phi^r$ *coincide on* $\ker(\pi^r)$*.*

Proof. We prove that $\ker(\pi^r) \cong M^{\otimes r}$ as $\mathcal{O}_X$-modules (with respect to either $\mathcal{O}_X$-module structure on $\mathcal{P}_\phi^r$). We proceed by induction on r.

For $r = 1$, $\mathcal{P}_\phi^1 = \mathcal{O}_X \oplus M$ and $\pi^1 : \mathcal{O}_X \oplus M \to \mathcal{O}_X$ is projection onto the first direct summand. It is clear that $\ker(\pi^1) = 0 \oplus M$. We compute that

$$\psi_{\mathrm{I}}^1(f) \cdot (0, m) = (f, 0) \cdot (0, m) = (0, fm) \qquad \text{and}$$
$$\psi_{\mathrm{II}}^1(f) \cdot (0, m) = (f, Df) \cdot (0, m) = (0, fm),$$

which shows that either of the $\mathcal{O}_X$ actions on $\ker(\pi^1)$ agrees with the natural $\mathcal{O}_X$ action on M. Therefore, $\ker(\pi^1) \cong M$ as $\mathcal{O}_X$-modules, and the statement is established for $r = 1$.

Suppose that the statement is true for $r - 1$.

By definition, $\pi^r = \mathrm{id}_{\mathcal{O}_X} \oplus (\pi^{r-1} \otimes \mathrm{id}_M)$, so that $\ker \pi^r = 0 \oplus (\ker(\pi^{r-1})\ {}_{\mathrm{II}}\!\otimes M)$. Let $U \subset X$ be open, $k \in \ker(\pi^{r-1})(U)$, $m \in M(U)$. We compute that

$$\psi_{\mathrm{I}}^r(f) \cdot (0, k \otimes m) = (f, 0) \cdot (0, k \otimes m) = (0, \pi^r((f, 0)) \cdot (k \otimes m) + 0) = (0, \big(\psi_{\mathrm{I}}^{r-1}(f) \cdot k\big) \otimes m) \text{ and}$$
$$\psi_{\mathrm{II}}^r(f) \cdot (0, k \otimes m) = (f, 1 \otimes Df) \cdot (0, k \otimes m) = (0, \pi^r((f, 1 \otimes Df)) \cdot (k \otimes m)) = (0, \big(\psi_{\mathrm{II}}^{r-1}(f) \cdot k\big) \otimes m).$$

By the inductive hypothesis, the ψ_{I}^{r-1} and ψ_{II}^{r-1} $\mathcal{O}_X$ actions coincide on $\ker(\pi^{r-1})$, and are isomorphic to the natural $\mathcal{O}_X$ action on $M^{\otimes r-1}$. Together with the above two computations, this shows that $\ker(\pi^r) \cong M^{\otimes r}$ as sheaves of abelian groups, and that the ψ_{I}^r and ψ_{II}^r $\mathcal{O}_X$ actions on $\ker(\pi^r)$ are isomorphic to the natural action on $M^{\otimes r}$. The statement follows for all r by induction. $\square$

Corollary 2.3. *The sequence of* $\mathcal{O}_X$*-modules*

$$0 \longrightarrow M^{\otimes r} \longrightarrow \mathcal{P}_\phi^r \xrightarrow{\pi^r} \mathcal{P}_\phi^{r-1} \longrightarrow 0 \tag{2.2}$$

is short exact (here $M^{\otimes r} \to \mathcal{P}_\phi^r$ is the inclusion of the kernel) with respect to either the ψ_{I} $\mathcal{O}_X$-module structures on $\mathcal{P}_\phi^r$ and $\mathcal{P}_\phi^{r-1}$, or the ψ_{II} $\mathcal{O}_X$-module structures on $\mathcal{P}_\phi^r$ and $\mathcal{P}_\phi^{r-1}$.

Proof. It remains to prove that $\pi^r : \mathcal{P}_\phi^r \to \mathcal{P}_\phi^{r-1}$ is surjective. This has a simple proof by induction on r. □

Corollary 2.4. *The sheaf $\mathcal{P}_\phi^r$ is locally free of rank $r+1$ with respect to either its ψ_{I}^r $\mathcal{O}_X$-module structure or its ψ_{II}^r $\mathcal{O}_X$-module structure.*

Proof. It is clear that $\mathcal{P}_\phi^0 = \mathcal{O}_X$ is a locally free $\mathcal{O}_X$-module of rank 1 with respect to either of its $\mathcal{O}_X$-module structures (in fact, the two structures coincide in this case). Since $M^{\otimes r}$ is locally free of rank 1, the corollary follows by induction on r using the short exact sequence (2.2). □

We record how the two $\mathcal{O}_X$-module structures on $\mathcal{P}_\phi^r$ look locally:

Proposition 2.5. *Let $U \subset X$ be open. For $f, g \in \mathcal{O}_X(U)$ and $t \in (\mathcal{P}_\phi^{r-1} {}_{\mathrm{II}}\!\otimes M)(U)$,*

$$\begin{aligned} \psi_{\mathrm{I}}^r(f) \cdot (g,t) &= (fg,\, \psi_{\mathrm{I}}^{r-1}(f)t) \qquad \textit{and} \\ \psi_{\mathrm{II}}^r(f) \cdot (g,t) &= (fg,\, \psi_{\mathrm{II}}^{r-1}(f)t + \pi^r(g,t) \otimes Df). \end{aligned}$$

Proof. Both computations follow immediately from the definitions (and the facts that $\psi_{\mathrm{I}}^{r-1} = \pi^r \circ \psi_{\mathrm{I}}^r$ and $\psi_{\mathrm{II}}^{r-1} = \pi^r \circ \psi_{\mathrm{II}}^r$ (Proposition 2.1)). □

Remark. The sheaves $\mathcal{P}_\phi^r$ will not in general be studied with the natural $\mathcal{O}_X$-module structure induced on the direct sum (the one given by $f \cdot (g \oplus (s \otimes m)) = (fg) \oplus ((\psi_{\mathrm{II}}^{r-1}(f)s) \otimes m)$ for $f, g \in \mathcal{O}_X(U)$, $s \in \mathcal{P}_\phi^{r-1}(U)$, $m \in M(U)$), as this structure seems to not have a good interpretation in terms of the derivation D and its iterates.

Filtration. In this paragraph, we construct a filtration of $\mathcal{P}_\phi^r$ by vector bundles.

Let r and k be integers with $r \geq k \geq 1$. Denote by $\pi^{r,k}$ the composition of projections

$$\pi^{r,k} := \pi^k \circ \pi^{k+1} \circ \cdots \circ \pi^{r-1} \circ \pi^r : \mathcal{P}^r_\phi \to \mathcal{P}^{k-1}_\phi.$$

The map $\pi^{r,k}$ is a morphism of $\mathcal{O}_X$-algebras (where $\mathcal{P}^r_\phi$ and $\mathcal{P}^{k-1}_\phi$ are either both ψ_{I}-modules or both ψ_{II}-modules). Define

$$\mathcal{N}_{r,k} := \ker(\pi^{r,k}).$$

The $\mathcal{O}_X$-module $\mathcal{N}_{r,k}$ fits into the exact sequence

$$0 \longrightarrow \mathcal{N}_{r,k} \longrightarrow \mathcal{P}^r_\phi \xrightarrow{\pi^{r,k}} \mathcal{P}^{k-1}_\phi \longrightarrow 0\ ,$$

and is therefore locally free of rank $r+1-k$.

Also, define

$$\mathcal{N}_{r,k} := 0 \text{ for } k \geq r+1 \qquad \text{and} \qquad \mathcal{N}_{r,k} := \mathcal{P}^r_\phi \ \text{ for } k \leq 0.$$

For $r-1 \geq k \geq 1$, we have the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{N}_{r,k+1} \arrow[d] \arrow[dl, dashed, "\exists"'] & & \\
0 \arrow[r] & \mathcal{N}_{r,k} \arrow[r] & \mathcal{P}^r_\phi \arrow[r] \arrow[d] & \mathcal{P}^{k-1}_\phi \arrow[r] & 0 \\
 & & \mathcal{P}^k_\phi \arrow[d] \arrow[ur] & & \\
 & & 0 & &
\end{tikzcd}$$

Therefore, we have the inclusions

$$
\begin{array}{ccccccccccccc}
\dots \longrightarrow & \mathcal{N}_{r,r+1} & \longrightarrow & \mathcal{N}_{r,r} & \longrightarrow & \mathcal{N}_{r,r-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{N}_{r,1} & \longrightarrow & \mathcal{N}_{r,0} & \longrightarrow \cdots \\
& \| & & \| & & & & & & & & \| & \\
& 0 & & M^{\otimes r} & & & & & & & & \mathcal{P}^r_\phi &
\end{array}
$$

and so $(\mathcal{N}_{r,k})_{k\in\mathbb{Z}}$ is a decreasing filtration of $\mathcal{P}^r_\phi$.

Now, suppose that $r' \geq r \geq k \geq 1$. We have the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{N}_{r',k} \arrow[r] \arrow[d, dashed, "\exists"'] & \mathcal{P}^{r'}_\phi \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \mathcal{N}_{r,k} \arrow[r] & \mathcal{P}^r_\phi \arrow[r] & \mathcal{P}^{k-1}_\phi \arrow[r] & 0
\end{tikzcd}
$$

Let $r \geq 0$ and j, k be integers. Identify the subsheaves $\mathcal{N}_{r,j}$ with their images in $\mathcal{P}^r_\phi$. We would like to establish that the filtration $(\mathcal{N}_{r,k})_{k\in\mathbb{Z}}$ is multiplicative, in the sense that

$$
\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} \subset \mathcal{N}_{r,j+k}, \tag{2.3}
$$

where $\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k}$ denotes the product of ideal sheaves in $\mathcal{P}^r_\phi$.

We begin by establishing the following lemma:

Lemma 2.6. *Let $r \geq 1$ and k be integers. Let $U \subset X$ be open, and suppose that $(f, s \otimes m)$ is a section in $\mathcal{N}_{r,k}(U) \subset \mathcal{P}^r_\phi(U)$ with $m \neq 0$ (here $f \in \mathcal{O}_X(U)$, $s \in \mathcal{P}^{r-1}_\phi(U)$, $m \in M(U)$). Then s is a section in $\mathcal{N}_{r-1,k-1}(U)$.*

Proof. Suppose first that $r \geq k \geq 2$. We may assume that U is affine. Chasing $(f, s \otimes m)$ through the commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{P}^r_\phi & \xrightarrow{\quad\quad\pi^r\quad\quad} & \mathcal{P}^{r-1}_\phi & \xrightarrow{\pi^{r-1}} & \cdots & \xrightarrow{\pi^k} & \mathcal{P}^{k-1}_\phi \\
\| & & \| & & & & \| \\
\mathcal{O}_X \oplus \left(\mathcal{P}^{r-1}_\phi {}_{\text{II}}\!\otimes M\right) & \to & \mathcal{O}_X \oplus \left(\mathcal{P}^{r-2}_\phi {}_{\text{II}}\!\otimes M\right) & \to & \cdots & \to & \mathcal{O}_X \oplus \left(\mathcal{P}^{k-2}_\phi {}_{\text{II}}\!\otimes M\right)
\end{array}
$$

we see that $\pi^{r,k}((f, s \otimes m)) = (f, \pi^{r-1,k-1}(s) \otimes m)$. Since $M(U)$ is a flat module (it is projective) and $m \neq 0$, we have $\pi^{r-1,k-1}(s) \otimes m = 0$ if and only if $\pi^{r-1,k-1}(s) = 0$, which shows that $s \in \mathcal{N}_{r-1,k-1}(U)$.

The cases when $k \leq 1$ and when $k > r$ are immediate. □

We proceed to show that the filtration of $\mathcal{P}^r_\phi$ by $\mathcal{N}_{r,k}$ is multiplicative.

We first show that $\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} = 0$ whenever $r < j + k$. We proceed by induction on r. The base case $r = 0$ is immediate. Suppose now toward induction that $\mathcal{N}_{r-1,j} \cdot \mathcal{N}_{r-1,k} = 0$ for all integers j, k with $r - 1 < j + k$. We show that then $\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} = 0$ for all integers j, k with $r < j + k$.

Suppose first that $j \geq 1$ and $k \geq 1$. Let $U \subset X$ be open, and let $(f, s \otimes m) \in \mathcal{N}_{r,j}(U)$ and $(g, t \otimes n) \in \mathcal{N}_{r,k}(U)$.[3] Their product is

$$\begin{aligned}
(f, s \otimes m) \cdot (g, t \otimes n) &= (fg, \pi^r((f, s \otimes m))(t \otimes n) + \pi^r((g, t \otimes n))(s \otimes m)) \\
&= (fg, ((\pi^r((f, s \otimes m))t) \otimes n) + ((\pi^r((g, t \otimes n))s) \otimes m)).
\end{aligned}$$

Since $j \geq 1$ and $k \geq 1$, we have $f = 0$ and $g = 0$, respectively. Since (as was shown above) $\pi^r(\mathcal{N}_{r,j}) = \mathcal{N}_{r-1,j}$ and $\pi^r(\mathcal{N}_{r,k}) = \mathcal{N}_{r-1,k}$, we have $\pi^r((f, s \otimes m)) \in \mathcal{N}_{r-1,j}(U)$ and $\pi^r((g, t \otimes n)) \in \mathcal{N}_{r-1,k}(U)$. Finally, by the lemma, if $m \neq 0$ then $s \in \mathcal{N}_{r-1,j-1}(U)$, and if $n \neq 0$ then $t \in \mathcal{N}_{r-1,k-1}(U)$.

Therefore,

$$\begin{array}{lllll}
\text{either} \quad \pi^r((f, s \otimes m))t & \text{is contained in} & \mathcal{N}_{r-1,j} \cdot \mathcal{N}_{r-1,k-1} = 0 & \text{or} & n = 0, \quad \text{and} \\
\text{either} \quad \pi^r((g, t \otimes n))s & \text{is contained in} & \mathcal{N}_{r-1,k} \cdot \mathcal{N}_{r-1,j-1} = 0 & \text{or} & m = 0,
\end{array}$$

and so $(f, s \otimes m) \cdot (g, t \otimes n) = 0$. These computations show that $\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} = 0$ in the case when $j \geq 1$ and $k \geq 1$. In the remaining cases, one of j, k is ≤ 0, hence the other is $> r$, so $\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} = 0$.

[3] Here $f, g \in \mathcal{O}_X(U)$, $s, t \in \mathcal{P}^{r-1}_\phi(U)$, $m, n \in M(U)$.

We now show that $\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} \subset \mathcal{N}_{r,j+k}$ for any integers $r \geq 0$ and j, k with $r \geq j+k$. Suppose first that $j \geq 1$ and $k \geq 1$. We have the commutative diagram

$$\begin{tikzcd}
 & \mathcal{P}_\phi^r \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{P}_\phi^{j-1} & \mathcal{P}_\phi^{j+k-1} \arrow[l] \arrow[r] & \mathcal{P}_\phi^{k-1}
\end{tikzcd}$$

It was shown above that $\pi^{r,j+k}(\mathcal{N}_{r,j}) = \mathcal{N}_{j+k-1,j}$ and $\pi^{r,j+k}(\mathcal{N}_{r,k}) = \mathcal{N}_{j+k-1,k}$. Since $\pi^{r,j+k}$ is a morphism of sheaves of rings, $\pi^{r,j+k}(\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k}) = \pi^{r,j+k}(\mathcal{N}_{r,j}) \cdot \pi^{r,j+k}(\mathcal{N}_{r,k}) = \mathcal{N}_{j+k-1,j} \cdot \mathcal{N}_{j+k-1,k}$. But since $j+k-1 < j+k$, it follows by the previous case that $\mathcal{N}_{j+k-1,j} \cdot \mathcal{N}_{j+k-1,k} = 0$. Therefore, $\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} \subset \mathcal{N}_{r,j+k}$ when $j \geq 1$ and $k \geq 1$. In the remaining cases, at least one of j, k is ≤ 0 and the result is immediate since the filtration is decreasing.

This establishes that the filtration of $\mathcal{P}_\phi^r$ by $\mathcal{N}_{r,k}$ is multiplicative.

In summary:

Theorem 2.7. *The sequence of vector bundles $(\mathcal{N}_{r,k})_{k \in \mathbb{Z}}$ constructed above is an (exhaustive and separated) decreasing multiplicative filtration of $\mathcal{P}_\phi^r$, in the sense that*

$$\mathcal{N}_{r,j} \supset \mathcal{N}_{r,k} \qquad \textit{for any integers } j \leq k \qquad \textit{and}$$

$$\mathcal{N}_{r,j} \cdot \mathcal{N}_{r,k} \subset \mathcal{N}_{r,j+k} \qquad \textit{for any integers } j, k.$$

Remark. It is immediate from the above that the associated graded algebra of the filtration of $\mathcal{P}_\phi^r$ by $\mathcal{N}_{r,k}$ is isomorphic to $\mathcal{O}_X \oplus M \oplus (M \otimes M) \oplus \cdots \oplus M^{\otimes r}$.

Differential operators. Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules, and let $r \geq 0$ be an integer. We recall ([Gro67, §16.8]) that a differential operator $\psi \colon \mathcal{F} \to \mathcal{G}$ of order $\leq r$ may be defined inductively as follows. Define a map ψ to be a differential operator of order ≤ 0 if ψ is an $\mathcal{O}_X$-linear. Then, supposing that a differential operator of order $\leq (r-1)$ has been defined, define a map ψ to be a differential operator of order $\leq r$ if for every pair of open sets $V \subset U$

in X and every function f in $\mathcal{O}_X(U)$, the map $\psi_f\colon \mathcal{F}|_U \to \mathcal{G}|_U$ given by

$$\psi_f(s) = \psi(fs) - f\psi(s) \qquad \text{for any } s \in \mathcal{F}(V)$$

is a differential operator of order $\le (r-1)$.

Theorem 2.8. *Let $r \ge k \ge 0$ be integers. Let t be a global section of $\mathcal{N}_{r,r-k}$.*

(i) The map $f \in \mathcal{O}_X(U) \mapsto \psi^r_{\mathrm{II}}(f)\,t$ is a differential operator of order $\le k$ with respect to the ψ^r_{I} $\mathcal{O}_X$-module structure of $\mathcal{P}^r_\phi$.

(ii) The map $f \in \mathcal{O}_X(U) \mapsto \psi^r_{\mathrm{I}}(f)\,t$ is a differential operator of order $\le k$ with respect to the ψ^r_{II} $\mathcal{O}_X$-module structure of $\mathcal{P}^r_\phi$.

Proof. We begin by observing that because $\pi^{r,1} \circ \psi^r_{\mathrm{I}} = \pi^{r,1} \circ \psi^r_{\mathrm{II}} = \mathrm{id}_{\mathcal{O}_X}$, $(\psi^r_{\mathrm{I}} - \psi^r_{\mathrm{II}})(U)(f)$ is in $\mathcal{N}_{r,1}(U)$ for any $U \subset X$ open and any $f \in \mathcal{O}_X(U)$.

(i) We proceed by induction on k. We first establish the case $k = 0$. Suppose that t is in $\mathcal{N}_{r,r}(X) = M^{\otimes r}(X)$:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}_{r,r} \arrow[r] \arrow[d, equal] & \mathcal{P}^r_\phi \arrow[r, "\pi^r"] & \mathcal{P}^{r-1}_\phi \arrow[r] & 0 \\ & M^{\otimes r} & & & \end{tikzcd}$$

It is required to show that the map $f \mapsto \psi^r_{\mathrm{II}}(f)\,t$ is $\mathcal{O}_X$-linear with respect to the ψ^r_{I} action. But by Proposition 2.2, the ψ^r_{I} and ψ^r_{II} actions coincide on $\mathcal{N}_{r,r}$, so that for any $g \in \mathcal{O}_X(U)$,

$$\psi^r_{\mathrm{II}}(gf)t = \psi^r_{\mathrm{II}}(g)(\psi^r_{\mathrm{II}}(f)t) = \psi^r_{\mathrm{I}}(g)(\psi^r_{\mathrm{II}}(f)t),$$

since $\psi^r_{\mathrm{II}}(f)t$ is in $\mathcal{N}_{r,r}(U)$. Therefore, $f \mapsto \psi^r_{\mathrm{II}}(f)\,t$ is $\mathcal{O}_X$-linear (and hence a differential operator of order ≤ 0).

Now suppose that the map $f \mapsto \psi^r_{\mathrm{II}}(f)t$ is a differential operator of order $\le (k-1)$

whenever $t \in \mathcal{N}_{r,r-(k-1)}$. Let $V \subset U \subset X$, $f \in \mathcal{O}_X(U)$, $s \in \mathcal{O}_X(V)$. We compute that

$$\begin{aligned}\psi^r_{\mathrm{II}}(fs)n - f \cdot_{\psi^r_{\mathrm{I}}} \psi^r_{\mathrm{II}}(s)t &= \psi^r_{\mathrm{II}}(f)\psi^r_{\mathrm{II}}(s)t - \psi^r_{\mathrm{I}}(f)\psi^r_{\mathrm{II}}(s)t \\ &= \psi^r_{\mathrm{II}}(s)\left(\psi^r_{\mathrm{II}}(f) - \psi^r_{\mathrm{I}}(f)\right)t.\end{aligned}$$

The product $(\psi^r_{\mathrm{II}}(f) - \psi^r_{\mathrm{I}}(f))\,t$ is in $\mathcal{N}_{r,1} \cdot \mathcal{N}_{r,r-k} \subset \mathcal{N}_{r,r-(k-1)}$. The map $s \mapsto \psi^r_{\mathrm{II}}(fs)t - f\psi^r_{\mathrm{II}}(s)t$ therefore has the form $s \mapsto \psi^r_{\mathrm{II}}(s)t'$ with $t' \in \mathcal{N}_{r,r-(k-1)}$; by the inductive hypothesis, this map is a differential operator of order $\leq (k-1)$. Therefore, the map $f \in \mathcal{O}_X(U) \mapsto \psi^r_{\mathrm{II}}(f)\,t$ is a differential operator of order $\leq k$.

(ii) The proof is similar to the proof of (i) (switching the roles of ψ^r_{I} and ψ^r_{II}). □

As an immediate corollary, we obtain:

Corollary 2.9. *The map $\psi^r_{\mathrm{II}} : \mathcal{O}_X \to \mathcal{P}^r_\phi$ is a differential operator of order $\leq r$ with respect to the ψ^r_{I}-module structure of $\mathcal{P}^r_\phi$; reciprocally, the map $\psi^r_{\mathrm{I}} : \mathcal{O}_X \to \mathcal{P}^r_\phi$ is a differential operator of order $\leq r$ with respect to the ψ^r_{II}-module structure of $\mathcal{P}^r_\phi$.*

Proof. The first statement is implied by part (i) of Theorem 2.8, and the second statement is implied by part (ii), both with $t = 1_{\mathcal{P}^r_\phi}$ (which is a global section of $\mathcal{N}_{r,0}$). □

We remind the reader that there exists a differential operator $d^r : \mathcal{O}_X \to \mathcal{P}^r$ of order r, such that the pair $(\mathcal{P}^r, d^r)$ satisfies the following universal property ([Gro67, §16.8]): for any $\mathcal{O}_X$-module $\mathcal{G}$, and any differential operator $\psi : \mathcal{O}_X \to \mathcal{G}$ of order $\leq r$, there exists a unique $\mathcal{O}_X$-linear morphism $\mathcal{P}^r \to \mathcal{G}$ that makes the following triangle commute:

$$\begin{tikzcd} \mathcal{P}^r \arrow[r, "\exists!"] & \mathcal{G} \\ \mathcal{O}_X \arrow[u, "d^r"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Therefore, as an immediate consequence of Corollary 2.9, we obtain $\mathcal{O}_X$-linear morphisms

$\mathcal{P}^r \to \mathcal{P}^r_\phi$ making the following triangle commute:

$$\begin{tikzcd} \mathcal{P}^r \arrow[r, "\exists!"] & \mathcal{P}^r_\phi \\ \mathcal{O}_X \arrow[u, "d^r"] \arrow[ur, "\psi^r_{\mathrm{II}}"'] & \end{tikzcd}$$

Let Δ be the diagonal of $X \times X$, and let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal. Let π_1 be the projection from $X \times X$ to the first factor (we are following the usual convention that $\mathcal{P}^r = (\pi_1)_* \left(\mathcal{O}_{X\times X}/\mathcal{I}_\Delta^{r+1}\right)$, cf. [Gro67, §16.3, 16.7]). Then for any $r \geq k-1 \geq 0$, we have a short exact sequence

$$0 \longrightarrow (\pi_1)_* \mathcal{I}_\Delta^k/\mathcal{I}_\Delta^{r+1} \longrightarrow \mathcal{P}^r \longrightarrow \mathcal{P}^{k-1} \longrightarrow 0$$

and consequently, by the uniqueness of the induced maps $\mathcal{P}^r \to \mathcal{P}^r_\phi$, we have the following commutative diagram with exact rows

(2.4)
$$\begin{tikzcd} 0 \arrow[r] & (\pi_1)_* \mathcal{I}_\Delta^k/\mathcal{I}_\Delta^{r+1} \arrow[r] \arrow[dd] & \mathcal{P}^r \arrow[rr] \arrow[dd] & & \mathcal{P}^{k-1} \arrow[r] \arrow[dd] & 0 \\ & & & \mathcal{O}_X \arrow[ul, "d^r"'] \arrow[ur, "d^{k-1}"] \arrow[dl, "\psi^r_{\mathrm{II}}"] \arrow[dr, "\psi^{k-1}_{\mathrm{II}}"'] & & \\ 0 \arrow[r] & \mathcal{N}_{r,k} \arrow[r] & \mathcal{P}^r_\phi \arrow[rr] & & \mathcal{P}^{k-1}_\phi \arrow[r] & 0 \end{tikzcd}$$

We conclude this paragraph with two minor asides.

Remarks. (i) In the special case of (2.4) when $k-1 = r-1$, we have a surjection $\mathrm{Sym}^r\left((\pi_1)_*\mathcal{I}_\Delta/\mathcal{I}_\Delta^2\right) \to (\pi_1)_*\mathcal{I}_\Delta^r/\mathcal{I}_\Delta^{r+1}$, as well as identifications $(\pi_1)_*\mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \cong \Omega_X$ and $\mathcal{N}_{r,r} \cong M^{\otimes r} = \mathrm{Sym}^r(M)$. We thereby obtain a map $\mathrm{Sym}^r(\Omega_X) \to \mathrm{Sym}^r(M)$, which may be shown (by a laborious inductive computation) to be equal to $r!\,\mathrm{Sym}^r(\phi)$, where $\phi\colon \Omega_X \to M$ is the defining derivation of $\mathcal{P}^r_\phi$.

(ii) In general, the vector bundle $\mathcal{P}^r_\phi$ does not split into a direct sum of line bundles. However, fixing one of the $\mathcal{O}_X$-module structures on $\mathcal{P}^r_\phi$, there is an isomorphism (of $\mathcal{O}_X$-modules) of $\mathcal{P}^r_\phi$ and the direct sum $\mathcal{O}_X \oplus (\mathcal{P}^{r-1}_\phi \,{}_{\mathrm{II}}\!\otimes M)$ (in general, the isomorphism depends on the choice of $\mathcal{O}_X$-module structure). For the ψ_{I} structure, the identity map gives the desired isomorphism. For the ψ_{II} structure, since the diagram

$$\begin{tikzcd} & & \mathcal{O}_X \arrow[d, "\psi^r_{\mathrm{II}}"'] \arrow[dr, "\mathrm{id}_{\mathcal{O}_X}"] & & \\ 0 \arrow[r] & \mathcal{P}^{r-1}_{\phi}\ {}_{\mathrm{II}}\!\otimes M \arrow[r] & \mathcal{P}^r_{\phi} \arrow[r, "\pi^{r,0}"'] & \mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

of $\mathcal{O}_X$-modules commutes, it follows by the splitting lemma that the map $\mathcal{P}^r_{\phi} \to \mathcal{O}_X \oplus (\mathcal{P}^{r-1}_{\phi}\ {}_{\mathrm{II}}\!\otimes M)$ given by $\pi^{r,1} \oplus \left(\mathrm{id}_{\mathcal{P}^r_{\phi}} - \psi^r_{\mathrm{II}} \circ \pi^{r,1}\right)$ is an isomorphism. On elements, this isomorphism may be described as follows: for $U \subset X$ open, $f \in \mathcal{O}_X(U)$ and $s \in (\mathcal{P}^{r-1}_{\phi}\ {}_{\mathrm{II}}\!\otimes M)(U)$, the isomorphism is $(f, s) \mapsto (f, s - (1 \otimes Df))$.

$\mathcal{O}_X$-modules with $\mathcal{P}^r_{\phi}$ coefficients. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For any $r \geq 0$, we define

$$\mathcal{P}^r_{\phi}(\mathcal{F}) := \mathcal{P}^r_{\phi}\ {}_{\mathrm{II}}\!\otimes \mathcal{F}. \tag{2.5}$$

(The symbol ${}_{\mathrm{II}}\!\otimes$ again indicates that the $\mathcal{O}_X$-module structure taken on $\mathcal{P}^r_{\phi}$ is the one induced by the map ψ^r_{II}.)

Define a morphism of sheaves of abelian groups

$$\mathcal{F} \to \mathcal{P}^r_{\phi}(\mathcal{F}) \qquad \text{by} \qquad \begin{aligned} \mathcal{F}(U) &\to \mathcal{P}^r_{\phi}(\mathcal{F})(U) \\ s &\mapsto 1_{\mathcal{P}^r_{\phi}} \otimes s \end{aligned} \qquad \text{for } U \subset X \text{ open.} \tag{2.6}$$

Since the triangle

$$\begin{tikzcd} \mathcal{P}^r(\mathcal{F}) \arrow[r, equal] & \mathcal{P}^r \otimes \mathcal{F} \arrow[r] & \mathcal{P}^r_{\phi}\ {}_{\mathrm{II}}\!\otimes \mathcal{F} \arrow[r, equal] & \mathcal{P}^r_{\phi}(\mathcal{F}) \\ & \mathcal{F} \arrow[u] \arrow[ur] & & \end{tikzcd}$$

commutes, $\mathcal{F} \to \mathcal{P}^r_{\phi}(\mathcal{F})$ is a differential operator of order $\leq r$.

The sheaf $\mathcal{P}^r_{\phi}(\mathcal{F})$ is a $\mathcal{P}^r_{\phi}$-module in a natural way. For each pair of integers r, k, define (cf. [Del71, 1.1.12])

$$\mathcal{N}_{r,k}(\mathcal{F}) := \mathrm{Im}\left(\mathcal{N}_{r,k}\ {}_{\mathrm{II}}\!\otimes \mathcal{F} \to \mathcal{P}^r_{\phi}\ {}_{\mathrm{II}}\!\otimes \mathcal{F}\right).$$

The $\mathcal{N}_{r,k}(\mathcal{F})$ are again $\mathcal{P}^r_\phi$-modules, and the collection $(\mathcal{N}_{r,k}(\mathcal{F}))_{k\in\mathbb{Z}}$ is a decreasing filtration of $\mathcal{P}^r_\phi(\mathcal{F})$ with the property $\mathcal{N}_{r,j}\cdot\mathcal{N}_{r,k}(\mathcal{F})\subset\mathcal{N}_{r,j+k}(\mathcal{F})$ for any r,j,k.

Tensoring the diagram (2.4) by $\mathcal{F}$, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & (\pi_1)_*\mathcal{I}^k_\Delta/\mathcal{I}^{r+1}_\Delta\ {}_{\pi_2}\!\otimes\mathcal{F} \arrow[r] \arrow[dd] & \mathcal{P}^r(\mathcal{F}) \arrow[rr] \arrow[dd] & & \mathcal{P}^{k-1}(\mathcal{F}) \arrow[r] \arrow[dd] & 0 \\
 & & & \mathcal{F} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & & \\
0 \arrow[r] & \mathcal{N}_{r,k}(\mathcal{F}) \arrow[r] & \mathcal{P}^r_\phi(\mathcal{F}) \arrow[rr] & & \mathcal{P}^{k-1}_\phi(\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

Expansion of sections. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $V\subset H^0(X,\mathcal{F})$ be a subspace of the global sections of $\mathcal{F}$ of dimension n. Let $V\otimes_k\mathcal{O}_X$ denote the sheaf

$$(V\otimes_k\mathcal{O}_X)(U)=V\otimes_k(\mathcal{O}_X(U))\qquad\text{for } U\subset X \text{ open.}$$

Any choice of basis for V as a k-vector space yields an isomorphism of $V\otimes_k\mathcal{O}_X$ with $\mathcal{O}_X^n$, so $V\otimes_k\mathcal{O}_X$ is a locally free sheaf of rank n. (The subscript on $\otimes_k$ will frequently be omitted from now on.)

For any $r\geq 0$, define a morphism of sheaves $\Psi^r_V: V\otimes\mathcal{O}_X\to\mathcal{P}^r_\phi(\mathcal{F})$ by

$$\begin{aligned}\Psi^r_V(U):(V\otimes\mathcal{O}_X)(U)&\to\mathcal{P}^r_\phi(\mathcal{F})(U)\\ s\otimes f&\mapsto f\cdot\Psi(U)\left(s|_U\right),\end{aligned}$$

where $\Psi:\mathcal{F}\to\mathcal{P}^r_\phi(\mathcal{F})$ is the differential operator (2.6).

Sometimes, by abuse of notation, this map will be denoted by

$$V\to\mathcal{P}^r(\mathcal{F}).$$

Inverse limit. The $\mathcal{O}_X$-modules $\mathcal{P}^r_\phi$ fit into an inverse system indexed by $\mathbb{N}$ (with its usual total order):

$$\cdots \xrightarrow{\pi^{r+2}} \mathcal{P}^{r+1}_\phi \xrightarrow{\pi^{r+1}} \mathcal{P}^r_\phi \xrightarrow{\pi^r} \mathcal{P}^{r-1}_\phi \xrightarrow{\pi^{r-1}} \cdots \xrightarrow{\pi^2} \mathcal{P}^1_\phi \xrightarrow{\pi^1} \mathcal{P}^0_\phi \ .$$

We denote the (inverse/projective) limit of the system by

$$\mathcal{P}^\infty_\phi := \lim_r \mathcal{P}^r_\phi \tag{2.7}$$

and denote the projection $\mathcal{P}^\infty_\phi \to \mathcal{P}^{r-1}_\phi$ by $\pi^{\infty,r}$.

Since the maps ψ^r_{I} and ψ^r_{II} are compatible with π^r (Proposition 2.1), there are induced maps of $\mathcal{O}_X$-modules

$$\psi^\infty_{\mathrm{I}} : \mathcal{O}_X \to \mathcal{P}^\infty_\phi \qquad \text{and} \qquad \psi^\infty_{\mathrm{II}} : \mathcal{O}_X \to \mathcal{P}^\infty_\phi.$$

We also define $\mathcal{P}^\infty_\phi(\mathcal{F}) := \lim_r \left(\mathcal{P}^r_\phi(\mathcal{F})\right)$.

2.3 Computations in local coordinates

Continuing with notation of the previous section, let X be a scheme, M be a line bundle on X, and $\phi\colon \Omega_X \to M$ be a morphism of $\mathcal{O}_X$-modules. (Denote by $D = \phi \circ d$ the derivation induced by ϕ.)

Throughout this section, $\mathcal{P}^r_\phi$ will be considered with its ψ^r_{I}-induced $\mathcal{O}_X$-module structure.

Let $U \subset X$ be open, and $s\colon U \to M$ be a local frame for M over U (that is, a nowhere-vanishing section of M over U). For every $r \geq 0$, the local frame s induces a local frame for the vector bundle $\mathcal{P}^r_\phi$ over U, which we now describe.

For $\mathcal{P}^1_\phi = \mathcal{O}_X \oplus M$, choose the frame over U given by the following pair of sections:

$$s_{1,0} := 1 \oplus 0 \qquad \text{and} \qquad s_{1,1} := 0 \oplus s.$$

For $r \geq 2$, inductively construct a local frame for $\mathcal{P}_\phi^r = \mathcal{O}_X \oplus \left(\mathcal{P}_\phi^{r-1}\ {}_{\mathrm{II}}\!\otimes M\right)$ over U as follows:

$$\begin{aligned}
s_{r,0} &:= 1 \oplus 0, \\
s_{r,1} &:= 1\left(0 \oplus \left(s_{r-1,0} \otimes s\right)\right), \\
&\ \vdots \\
s_{r,k} &:= k\left(0 \oplus \left(s_{r-1,k-1} \otimes s\right)\right), \\
&\ \vdots \\
s_{r,r} &:= r\left(0 \oplus \left(s_{r-1,r-1} \otimes s\right)\right).
\end{aligned}$$

For example, for $r = 2$, one gets the following local frame for $\mathcal{P}_\phi^2 = \mathcal{O}_X \oplus \left(\left(\mathcal{O}_X \oplus M\right)\ {}_{\mathrm{II}}\!\otimes M\right)$:

$$s_{2,0} = 1 \oplus 0, \qquad s_{2,1} = 0 \oplus \left(\left(1 \oplus 0\right) \otimes s\right), \qquad s_{2,2} = 2\left(0 \oplus \left(\left(0 \oplus s\right) \otimes s\right)\right).$$

We also set $s_{0,0} = 1$, which is a global frame for $\mathcal{O}_X$. The inductive construction of the $s_{r,k}$ could have started with $s_{0,0}$.

Scaling the elements of the frame as above simplifies several expressions that appear later in this section. Here, we note that

$$\begin{aligned}
s_{r,r} &= r\big(0 \oplus \left(s_{r-1,r-1} \otimes s\right)\big) \\
&= r(r-1)(0 \oplus \left(\left(s_{r-2,r-2} \otimes s\right) \otimes s\right) \\
&= r!(0 \oplus \underbrace{((\cdots(}_{r \text{ parentheses}} \quad 0 \oplus s) \otimes s)\cdots) \otimes s))
\end{aligned}$$

It follows that under the identification of $M^{\otimes r}$ with $\ker \pi^r$, the induced frame $s^{\otimes r}$ of $M^{\otimes r}$ over the open set U is identified with the frame $s_{r,r}/r!$ of $\ker \pi^r \subset \mathcal{P}_\phi^r$.

We have (by induction on r)

$$\pi^r(s_{r,k}) = \begin{cases} s_{r-1,k}, & k \le r-1, \\ 0, & k = r, \end{cases} \qquad \text{and so} \qquad \pi^{r,j}(s_{r,k}) = \begin{cases} s_{j-1,k}, & k \le j-1, \\ 0, & j \le k \le r. \end{cases} \tag{2.8}$$

Next, we describe the maps ψ^r_{I}, ψ^r_{II} and π^r with respect to the frame $(s_{r,k})^r_{k=0}$ on $\mathcal{P}^r_\phi$.

Let $f \in \mathcal{O}_X(U)$. Applying the derivation $D(U)$ to f, we obtain a section of M over U. Define $D^1_s(f) \in \mathcal{O}_X(U)$ by

$$D(U)(f) =: D^1_s(f)s.$$

Then, for $r \ge 1$, define $D^{r+1}_s(f) \in \mathcal{O}_X(U)$ inductively by

$$D(U)(D^r_s(f)) =: D^{r+1}_s(f)s.$$

(Define $D^0_s(f) := f$.)

We record the following properties of the $D^r_s(\cdot)$ for future use. Both properties are not difficult to check by induction.

Lemma 2.10. *Let $f, g \in \mathcal{O}_X(U)$.*

(i) For any integers $r, r' \ge 1$, $D^r_s(D^{r'}_s(f)) = D^{r+r'}_s(f)$.

(ii) For any integer $r \ge 1$, $D^r_s(fg) = \sum_{j=0}^r \binom{r}{j} D^j_s(f) D^{r-j}_s(g)$.

In particular, $D^1_s(fg) = fD^1_s(g) + gD^1_s(f)$, so $D^1_s(\cdot)$ is a derivation on $\mathcal{O}_X(U)$.

As a consequence of (ii), we also have the higher-order Leibniz rule:

$$D^r_s(f_1 \cdots f_k) = \sum_{j_1+\cdots+j_k=r} \binom{r}{j_1, j_2, \dots, j_k} D^{j_1}_s(f_1) \cdots D^{j_k}_s(f_k), \qquad \text{where} \qquad \binom{r}{j_1, j_2, \dots, j_k} = \frac{r!}{j_1! j_2! \cdots j_k!}.$$

Introduce the following notation: for any $f_0, \dots, f_r \in \mathcal{O}_X(U)$,

$$(f_0,\, f_1, \dots, f_r)_s := f_0\, s_{r,0} + f_1\, s_{r,1} + \cdots + f_r\, s_{r,r} \in \mathcal{P}^r_\phi(U).$$

The set $\{(f_0,\, f_1, \dots, f_r)_s \colon f_0, \dots, f_r \in \mathcal{O}_X(U)\}$ is evidently a k-vector space.

Theorem 2.11. *Let $r \geq 1$ be an integer, $s \colon U \to M$ be a local frame for M over U, $s_{r,0}, \dots, s_{r,r}$ and $s_{r-1,0}, \dots, s_{r-1,r-1}$ be the induced local frames for $\mathcal{P}^r_\phi$ and $\mathcal{P}^{r-1}_\phi$, respectively, and f be an element of $\mathcal{O}_X(U)$.*

(i) $\psi^r_{\mathrm{I}}(U)(f) = (f, 0, \dots, 0)_s$.

(ii) $\psi^r_{\mathrm{II}}(U)(f) = \left(f,\ \dfrac{1}{1!}\, D^1_s(f),\ \dots, \dfrac{1}{k!}\, D^k_s(f),\ \dots, \dfrac{1}{r!}\, D^r_s(f)\right)_s$.

(iii) For $f_0, \dots, f_r \in \mathcal{O}_X(U)$, $\quad \pi^{r,k}\left(\,(f_0, f_1, \dots, f_{r-1}, f_r)_s\,\right) = (f_0, f_1, \dots, f_{k-1})_s \quad (\forall 1 \leq k \leq r)$.

Proof. (i) By definition, $\psi^r_{\mathrm{I}} = \mathrm{id}_{\mathcal{O}_X} \oplus 0$. Therefore,

$$\psi^r_{\mathrm{I}}(U)(f) = \mathrm{id}_{\mathcal{O}_X}(U)(f) \oplus 0 = f\,(1 \oplus 0) = f\, s_{r,0} = (f,\, 0,\, \dots,\, 0)_s.$$

(ii) We check the claim by induction. For $r = 1$, we have $\psi^1_{\mathrm{II}} = \mathrm{id}_{\mathcal{O}_X} \oplus D$, and

$$\begin{aligned}
\psi^1_{\mathrm{II}}(U)(f) &= \mathrm{id}_{\mathcal{O}_X}(U)(f) \oplus D(U)(f) \\
&= f \oplus D^1_s(f)s \\
&= f s_{1,0} + D^1_s(f) s_{1,1} \\
&= (f,\, D^1_s(f))_s,
\end{aligned}$$

verifying the claim in this case. Now, suppose that the claim is true for $r-1$. By definition, $\psi^r_{\mathrm{II}} = \mathrm{id}_{\mathcal{O}_X} \oplus (1 \otimes D)$. We have (applying the result of Lemma 2.10(i) to compute

$\psi_{\mathrm{II}}^{r-1}(D_s^1(f)))$

$$\begin{aligned}
\psi_{\mathrm{II}}^r(U)(f) &= \mathrm{id}_{\mathcal{O}_X}(U)(f) \oplus (1 \otimes D(U)(f)) \\
&= f \oplus \left(1 \otimes D_s^1(f)s\right) \\
&= f \oplus \left(\psi_{\mathrm{II}}^{r-1}(U)(D_s^1(f)) \otimes s\right) \\
&= f \oplus \left((D_s^1(f),\ \frac{1}{1!}\, D_s^2(f), \ldots,\ \frac{1}{(r-1)!}\, D_s^r(f))_s \otimes s\right) \\
&= f \oplus \left((D_s^1(f)\, s_{r-1,0} + \frac{D_s^2(f)}{1!}\, s_{r-1,1} + \cdots + \frac{D_s^r(f)}{(r-1)!}\, s_{r-1,r-1}) \otimes s\right) \\
&= (f,\ \frac{1}{1!}\, D_s^1(f),\ \frac{1}{2!}\, D_s^2(f), \ldots,\ \frac{1}{r!} D_s^r(f))_s,
\end{aligned}$$

and the claim follows for any r by induction.

(iii) By the observation recorded in (2.8), we have

$$\begin{aligned}
\pi^{r,k}\left((f_0, f_1, \ldots, f_{r-1}, f_r)_s\right) &= \pi^{r,k}\left(f_0\, s_{r,0} + \cdots + f_{r-1}\, s_{r,r-1} + f_r\, s_{r,r}\right) \\
&= f_0\, s_{k-1,0} + \cdots + f_{k-1}\, s_{k-1,k-1} + \underbrace{0 + \cdots + 0}_{k \text{ zeroes}} \\
&= (f_0, \ldots, f_{k-1})_s.
\end{aligned}$$

□

Multiplication in local coordinates. Let $U \subset X$ be open, and $s\colon U \to M$ be a local frame for M over U. We continue with the constructions and notation established earlier in this section, and interpret the product structure on $\mathcal{P}_\phi^r$ in local coordinates.

Let $f_0, f_1, \ldots, f_r, g_0, g_1, \ldots, g_r \in \mathcal{O}_X(U)$. Set $F_r(t) = f_0 + f_1 t + \cdots + f_r t^r \in k[t]/(t^{r+1})$ and $G_r(t) = g_0 + g_1 t + \cdots + g_r t^r \in k[t]/(t^{r+1})$. Define $h_k \in \mathcal{O}_X(U)$ by

$$F_r(t)G_r(t) = \sum_{k=0}^{r} h_k t^k \quad \in k[t]/(t^{r+1}).$$

More explicitly,

$$h_0 := f_0 g_0, \quad h_1 := f_0 g_1 + f_1 g_0, \quad \ldots, \quad h_k := \sum_{j=0}^{k} f_j g_{k-j}, \quad \ldots, \quad h_r := \sum_{j=0}^{r} f_j g_{r-j}.$$

Theorem 2.12. *With the above notation, the product in $\mathcal{P}^r_\phi(U)$ is given by*

$$(f_0, f_1, \ldots, f_r)_s \cdot_{\mathcal{P}^r_\phi(U)} (g_0, g_1, \ldots, g_r)_s = (h_0, h_1, \ldots, h_r)_s. \tag{2.9}$$

Proof. Proceed by induction on r. For $r = 1$, the multiplication is given by

$$(f_0 \oplus (f_1 s)) \cdot (g_0 \oplus (g_1 s)) = f_0 g_0 \oplus (f_0 g_1\, s + g_0 f_1\, s) = f_0 g_0 \oplus ((f_0 g_1 + f_1 g_0)\, s),$$

verifying the base case.

Suppose that the statement is true for $r - 1$. Set $\tilde{f}_k = k \cdot f_k$, $\tilde{g}_k = k \cdot g_k$ and $\tilde{h}_k = k \cdot h_k$. We compute that

$$\begin{aligned}
&(f_0, f_1, \ldots, f_r)_s \cdot_{\mathcal{P}^r_\phi(U)} (g_0, g_1, \ldots, g_r)_s \\
&= f_0 \oplus \left(\left(\tilde{f}_1, \ldots, \tilde{f}_r\right)_s \otimes s\right) \cdot_{\mathcal{P}^r_\phi(U)} g_0 \oplus ((\tilde{g}_1, \ldots, \tilde{g}_r)_s \otimes s) \\
&= (f_0 g_0) \oplus \left[\pi^r ((f_0, f_1, \ldots, f_r)_s) \cdot (\tilde{g}_1, \ldots, \tilde{g}_r)_s \otimes s + \pi^r ((g_0, g_1, \ldots, g_r)_s) \cdot \left(\tilde{f}_1, \ldots, \tilde{f}_r\right)_s \otimes s\right] \\
&= (f_0 g_0) \oplus \left[\left((f_0, f_1, \ldots, f_{r-1})_s \cdot (\tilde{g}_1, \ldots, \tilde{g}_r)_s + (g_0, g_1, \ldots, g_{r-1})_s \cdot \left(\tilde{f}_1, \ldots, \tilde{f}_r\right)_s\right) \otimes s\right].
\end{aligned}$$

By the inductive hypothesis,

$$\begin{aligned}
&(f_0, f_1, \ldots, f_{r-1})_s \cdot (\tilde{g}_1, \ldots, \tilde{g}_r)_s = (f_0 \tilde{g}_1, f_0 \tilde{g}_2 + f_1 \tilde{g}_1, \ldots, \sum_{j=0}^{r-1} f_j \tilde{g}_{r-j})_s \qquad \text{and} \\
&(g_0, g_1, \ldots, g_{r-1})_s \cdot \left(\tilde{f}_1, \ldots, \tilde{f}_r\right)_s = (g_0 \tilde{f}_1, g_0 \tilde{f}_2 + g_1 \tilde{f}_1, \ldots, \sum_{j=0}^{r-1} g_j \tilde{f}_{r-j})_s.
\end{aligned}$$

Since

$$\sum_{j=0}^{k-1} f_j \tilde{g}_{k-j} + g_j \tilde{f}_{k-j} = f_0 \tilde{g}_k + \left(\sum_{j=1}^{k-1} f_j \tilde{g}_{k-j} + \tilde{f}_j g_{k-j} \right) + \tilde{f}_k g_0 = \tilde{h}_k,$$

we have

$$(f_0, f_1, \ldots, f_r)_s \cdot_{\mathcal{P}_\phi^r(U)} (g_0, g_1, \ldots, g_r)_s = (f_0 g_0) \oplus \left(\left(\tilde{h}_1, \ldots, \tilde{h}_r\right)_s \otimes s \right) = (h_0, h_1, \ldots, h_r)_s,$$

and the statement is true for all r by induction. □

Remark. One of the motivations for defining the global product structure on $\mathcal{P}_\phi^r$ in the way we have defined it comes from the Leibniz rule. Set $\delta_s^k(f) = D_s^k(f)/k!$ $(0 \le k \le r)$. As computed above, for any $f, g \in \mathcal{O}_X(U)$,

$$\begin{aligned} \psi_{\mathrm{II}}^r(U)(fg) &= \left(D_s^0(fg), \ \frac{D_s^1(fg)}{1!}, \ldots, \ \frac{D_s^k(fg)}{k!}, \ldots, \ \frac{D_s^r(fg)}{r!} \right)_s \\ &= \left(D_s^0(f) D_s^0(g), \ \frac{D_s^0(f) D_s^1(g) + D_s^1(f) D_s^0(g)}{1!}, \ldots, \ \frac{\sum_{j=0}^r \binom{r}{j} D_s^j(f) D_s^{r-j}(g)}{r!} \right)_s \end{aligned}$$

Since

$$\sum_{j=0}^k \binom{k}{j} \frac{1}{r!} D_s^j(f) D_s^{k-j}(g) = \sum_{j=0}^k \frac{D_s^j(f)}{j!} \frac{D_s^{k-j}(g)}{(k-j)!} = \sum_{j=0}^k \delta_s^j(f)\, \delta_s^{k-j}(g),$$

the latter is equal to

$$\left(\delta_s^0(f)\, \delta_s^0(g), \ \delta_s^0(f)\, \delta_s^1(g) + \delta_s^1(f)\, \delta_s^0(g), \ldots, \sum_{j=0}^r \delta_s^j(f)\, \delta_s^{r-j}(g) \right)_s$$

On the other hand, we have

$$\begin{aligned} &\psi_{\mathrm{II}}^r(U)(f) \cdot_{\mathcal{P}_\phi^r(U)} \psi_{\mathrm{II}}^r(U)(g) \\ &= \left(\delta_s^0(f), \ \delta_s^1(f), \ldots, \ \delta_s^k(f), \ldots, \ \delta_s^r(f) \right)_s \cdot_{\mathcal{P}_\phi^r(U)} \left(\delta_s^0(g), \ \delta_s^1(g), \ldots, \ \delta_s^k(g), \ldots, \ \delta_s^r(g) \right)_s . \end{aligned}$$

Since the morphism $\psi^r_{\mathrm{II}}(U) : \mathcal{O}_X(U) \to \mathcal{P}^r_\phi(U)$ should preserve the product structure, the definition of the product on $\mathcal{P}^r_\phi$ should be such that the results of the two computations are equal.

The filtration in local coordinates. The vector bundles $\mathcal{N}_{r,k}$ have a simple local interpretation. Let $U \subset X$ be open, and let $s : U \to X$ be a local frame for M over U (and continue with the notation of the previous sections). The following proposition is immediate given previously proven information:

Proposition 2.13. *A section t of $\mathcal{P}^r_\phi(U)$ is in $\mathcal{N}_{r,k}(U)$ if and only if t has the form*

$$t = (\underbrace{0, \ldots, 0}_{k\ 0\text{'s}}, f_k, \ldots, f_r)_s \qquad \textit{for some } f_k, \ldots, f_r \in \mathcal{O}_X(U).$$

From the local point of view, it is particularly clear that the filtration of $\mathcal{P}^r_\phi$ by $\mathcal{N}_{r,k}$ is decreasing and multiplicative.

2.3.1 Transition functions for $\mathcal{P}^r_\phi$

For the convenience of the reader, we begin by setting down our notation for transition functions.

Let E be a locally free sheaf of rank $r+1$ over a scheme X. For an open subset $U \subset X$, a local frame for E over U is a collection $s_0, s_1, \ldots, s_r \in E(U)$ of sections of E that determines an isomorphism $\mathcal{O}_U^{\oplus r+1} \to E|_U$ of $\mathcal{O}_U$-modules by

$$\mathcal{O}_U^{\oplus r+1}(V) \xrightarrow{\quad (\, s_0|_V \;\; s_1|_V \;\; \cdots \;\; s_r|_V \,) \quad} E|_U\,(V)$$

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_r \end{pmatrix} \longmapsto \begin{pmatrix} s_0|_V & s_1|_V & \cdots & s_r|_V \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_r \end{pmatrix} = f_0\ s_0|_V + f_1\ s_1|_V + \cdots + f_r\ s_r|_V$$

for each open subset $V \subset U$.

Let U_α and U_β be open subsets of X, and suppose that $s_{\alpha,0}, \ldots, s_{\alpha,r}$ and $s_{\beta,0},\ s_{\beta,1},\ \ldots,\ s_{\beta,r}$ are local frames for E over U_α and U_β, respectively. Write $U_{\alpha\beta} := U_\alpha \cap U_\beta$ and suppose that $U_{\alpha\beta} \neq \varnothing$.

Then, we have

$$\begin{aligned} s_{\alpha,0}|_{U_{\alpha\beta}} &= \tau_{\alpha\beta,00}\ s_{\beta,0}|_{U_{\alpha\beta}} + \cdots + \tau_{\alpha\beta,0r}\ s_{\beta,r}|_{U_{\alpha\beta}} \\ &\ \ \vdots \\ s_{\alpha,r}|_{U_{\alpha\beta}} &= \tau_{\alpha\beta,r0}\ s_{\beta,0}|_{U_{\alpha\beta}} + \cdots + \tau_{\alpha\beta,rr}\ s_{\beta,r}|_{U_{\alpha\beta}} \end{aligned} \qquad \text{i.e.} \qquad \begin{pmatrix} s_{\alpha,0}|_{U_{\alpha\beta}} \\ \vdots \\ s_{\alpha,r}|_{U_{\alpha\beta}} \end{pmatrix} = \tau_{\alpha\beta} \begin{pmatrix} s_{\beta,0}|_{U_{\alpha\beta}} \\ \vdots \\ s_{\beta,r}|_{U_{\alpha\beta}} \end{pmatrix},$$

where $(\tau_{\alpha\beta})_{ij} := \tau_{\alpha\beta,ij},\ \tau_{\alpha\beta,ij} \in \mathcal{O}_X(U_{\alpha\beta})$ and $\tau_{\alpha\beta} \in \mathrm{GL}_{r+1}(U_{\alpha\beta})$. The matrix $\tau_{\alpha\beta}$ is called the transition function from the frame $(s_{\beta,j})_j$ to the frame $(s_{\alpha,j})_j$.

The matrices $\tau_{\alpha\beta}$ satisfy the identities $\tau_{\alpha\beta}\tau_{\beta\alpha} = \mathrm{id}$ on double intersections and $\tau_{\alpha\beta}\tau_{\beta\gamma}\tau_{\gamma\alpha} = \mathrm{id}$ on triple intersections.

In the two local frames above, a section $f \in E(U_{\alpha\beta})$ has two expansions $f = \sum_j f_{\alpha,j}\ s_{\alpha,j}|_{U_{\alpha\beta}} =$

$\sum_j f_{\beta,j}\, s_{\beta,j}|_{U_{\alpha\beta}}$ and one computes that

$$\begin{pmatrix} f_{\alpha,0} \\ f_{\alpha,1} \\ \vdots \\ f_{\alpha,r} \end{pmatrix} = (\tau'_{\alpha\beta})^{-1} \begin{pmatrix} f_{\beta,0} \\ f_{\beta,1} \\ \vdots \\ f_{\beta,r} \end{pmatrix} = \tau'_{\beta\alpha} \begin{pmatrix} f_{\beta,0} \\ f_{\beta,1} \\ \vdots \\ f_{\beta,r} \end{pmatrix},$$

where $'$ denotes the matrix transpose.

Continuing with the notation of the previous sections, let X be a scheme,[4] M be a line bundle over X, and $\phi\colon \Omega_X \to M$ be a morphism of $\mathcal{O}_X$-modules. Suppose that $s_\alpha\colon U_\alpha \to M$ and $s_\beta\colon U_\beta \to M$ are local frames for M over U_α and U_β, respectively, suppose that $U_{\alpha\beta} \neq \varnothing$, and let $\tau_{\alpha\beta} \in \mathcal{O}_X(U)^*$ be the transition function (so that $s_\beta|_{U_{\alpha\beta}} = \tau_{\alpha\beta}\, s_\alpha|_{U_{\alpha\beta}}$).

For $r \geq 0$, denote by $\tau_{r,\alpha\beta}$ the transition functions for the locally free sheaf $\mathcal{P}^r_\phi$ with respect to the induced frames $(s_\alpha)_{r,j}$ and $(s_\beta)_{r,j}$ on $\mathcal{P}^r_\phi$ over U_α and U_β, respectively, introduced in §2.3; the notation $D^k_s(f) \in \mathcal{O}_X(U)$ with respect to a choice of frame s of M over U (and hence a choice of induced frame on $\mathcal{P}^r_\phi$ over U) was also introduced in this section. With respect to the local frames s_α, s_β fixed in the previous paragraph, denote $D^k_{s_\alpha}(f)$ by $D^k_\alpha(f)$, and $D^k_{s_\beta}(f)$ by $D^k_\beta(f)$. By Theorem 2.11, we then have $\psi^r_{\mathrm{II}}(U_\alpha)(f) = \sum_j D^j_\alpha(f)\,(s_\alpha)_{r,j}$ and $\psi^r_{\mathrm{II}}(U_\alpha)(f) = \sum_j D^j_\beta(f)\,(s_\beta)_{r,j}$. Thus, on $U_{\alpha\beta}$,

$$\begin{pmatrix} D^0_\alpha(f) \\ D^1_\alpha(f) \\ \vdots \\ D^r_\alpha(f) \end{pmatrix} = \tau'_{r,\beta\alpha} \begin{pmatrix} D^0_\beta(f) \\ D^1_\beta(f) \\ \vdots \\ D^r_\beta(f) \end{pmatrix} \qquad \text{for all } f \in \mathcal{O}_X(U_{\alpha\beta}). \tag{2.10}$$

The purpose of Lemma 2.14 below is to prove a kind of converse to the above observation — namely, that if a family $\{\rho_{\alpha\beta} \in \mathrm{GL}_{r+1}(U_{\alpha\beta})\}$ satisfies equations (2.10), then in fact $\rho_{\alpha\beta}$

[4] We remind the reader that by convention all schemes considered in this paper are reduced and of finite type over an algebraically closed field of characteristic zero; in particular, all schemes are Noetherian.

are the transition functions for $\mathcal{P}^r_\phi$ (in other words, the transition functions are determined by the image of ψ_{II}). The case $\phi = 0$ makes it clear that some kind of non-vanishing of ϕ is necessary for such a statement to hold.

Let $X = X_1 \cup \cdots \cup X_k$ be the decomposition of X into irreducible components, and let η_i be the generic point of X_i. We define a morphism $\xi \colon E \to E'$ of vector bundles on X to be generically surjective if $\xi_{\eta_i} \colon E_{\eta_i} \to E'_{\eta_i}$ is surjective for each $i = 1, \ldots, k$.

Lemma 2.14. *Suppose that $\phi \colon \Omega_X \to M$ is generically surjective. Fix an integer $r \geq 0$. Let $s_\alpha \colon U_\alpha \to M$ and $s_\beta \colon U_\beta \to M$ be local frames of M, and suppose that $U_{\alpha\beta} \neq \varnothing$. Let $\tau_{\alpha\beta}$ be the transition function from s_β to s_α, and $\tau_{r,\alpha\beta}$ be the transition function from the s_β-induced frame on $\mathcal{P}^r_\phi$ over U_β to the s_α-induced frame on $\mathcal{P}^r_\phi$ over U_α.*

Suppose that $\rho_{\alpha\beta} \in \mathrm{GL}_{r+1}(U_{\alpha\beta})$ satisfies

$$\begin{pmatrix} D^0_\beta(f) \\ D^1_\beta(f) \\ \vdots \\ D^r_\beta(f) \end{pmatrix} = \rho_{\alpha\beta} \begin{pmatrix} D^0_\alpha(f) \\ D^1_\alpha(f) \\ \vdots \\ D^r_\alpha(f) \end{pmatrix} \qquad \text{for each } f \in \mathcal{O}_X(U_{\alpha\beta}). \tag{2.11}$$

Then $\tau'_{r,\alpha\beta} = \rho_{\alpha\beta}$.

Proof. We begin with a sequence of reductions. We will think of $\tau'_{r,\alpha\beta}$ and $\rho_{\alpha\beta}$ as invertible $(r+1) \times (r+1)$ matrices with entries in $\mathcal{O}_X(U_{\alpha\beta})$; the equality $\tau'_{r,\alpha\beta} = \rho_{\alpha\beta}$ can then be shown by checking that each entry of $\tau'_{r,\alpha\beta}$ is equal to the corresponding entry of $\rho_{\alpha\beta}$.

- *$U_{\alpha\beta}$ may be assumed to be an open affine.* Choose a cover of $U_{\alpha\beta}$ by open affines. If $\tau'_{r,\alpha\beta} = \rho_{\alpha\beta}$ over each piece of the cover, then $\tau'_{r,\alpha\beta} = \rho_{\alpha\beta}$ holds over $U_{\alpha\beta}$ (because $\mathcal{O}_X$ is a sheaf!).

- *X may be assumed to be irreducible.* If $U_{\alpha\beta}$ only intersects one irreducible component of X, there is nothing to prove. Otherwise, suppose that $U_{\alpha\beta}$ intersects the components $X_{i_1}, \ldots, X_{i_m}$. Since the map $\mathcal{O}_X \to \bigoplus \mathcal{O}_{X_{i_j}}$ is injective (as can be seen by considering the corresponding map $A \to \bigoplus A/\mathfrak{p}_{i_j}$ of rings), if equality $\tau'_{r,\alpha\beta} = \rho_{\alpha\beta}$ holds on each $U_{\alpha\beta} \cap X_{i_j}$, then it also holds on $U_{\alpha\beta}$.

- *X may be replaced by an open subset U that has a nonempty intersection with $U_{\alpha\beta}$.* Since X is reduced and irreducible, it is integral. Therefore, the restriction map $\mathcal{O}_X \to \mathcal{O}_U$ is injective (to see this, note that U can be covered by distinguished open affines, and (after restricting to a distinguished affine) the map of rings corresponding to the restriction $\mathcal{O}_X \to \mathcal{O}_U$ is the localization map $A \to A_f$, which is injective when A is an integral domain). It follows that, if equality $\tau'_{r,\alpha\beta} = \rho_{\alpha\beta}$ holds after restriction to U, then it also holds over X.

- *X may be assumed to be smooth.* Replace X by its open subset of smooth points.

- *ϕ may be assumed to be surjective.* Pass to the open subset of X over which ϕ is surjective.

- *It is enough to find a point $p \in U_{\alpha\beta}$, an open neighbourhood U of p, and $f_1, \ldots, f_{r+1}$ in $\mathcal{O}_X(U)$, such that*

$$\det \begin{pmatrix} f_1 & \cdots & f_{r+1} \\ D_\alpha(f_1) & \cdots & D_\alpha(f_{r+1}) \\ \vdots & \ddots & \vdots \\ D_\alpha^r(f_1) & \cdots & D_\alpha^r(f_{r+1}) \end{pmatrix} \neq 0.$$

Indeed, by equations (2.10) and (2.11), for any $f_1, \ldots, f_{r+1} \in \mathcal{O}_X(U_{\alpha\beta})$, we have

$$\rho_{\alpha\beta} \begin{pmatrix} f_1 & \cdots & f_{r+1} \\ D_\alpha(f_1) & \cdots & D_\alpha(f_{r+1}) \\ \vdots & \ddots & \vdots \\ D_\alpha^r(f_1) & \cdots & D_\alpha^r(f_{r+1}) \end{pmatrix} = \begin{pmatrix} f_1 & \cdots & f_{r+1} \\ D_\beta(f_1) & \cdots & D_\beta(f_{r+1}) \\ \vdots & \ddots & \vdots \\ D_\beta^r(f_1) & \cdots & D_\beta^r(f_{r+1}) \end{pmatrix} = \tau'_{r,\alpha\beta} \begin{pmatrix} f_1 & \cdots & f_{r+1} \\ D_\alpha(f_1) & \cdots & D_\alpha(f_{r+1}) \\ \vdots & \ddots & \vdots \\ D_\alpha^r(f_1) & \cdots & D_\alpha^r(f_{r+1}) \end{pmatrix}$$

so that

$$\left(\rho_{\alpha\beta} - \tau'_{r,\alpha\beta}\right)\begin{pmatrix} f_1 & \cdots & f_{r+1} \\ D^1_\alpha(f_1) & \cdots & D^1_\alpha(f_{r+1}) \\ \vdots & \ddots & \vdots \\ D^r_\alpha(f_1) & \cdots & D^r_\alpha(f_{r+1}) \end{pmatrix} = 0.$$

The reduction now follows by replacing X by U.

We proceed to find a collection of functions satisfying the conditions of the last reduction.

Let p be a point of $U_{\alpha\beta}$, $\mathcal{O}_{X,p}$ be the local ring of X at p, and $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$.

For $h_1, \ldots, h_k \in \mathcal{O}_X(U_\alpha)$, we have the higher-order Leibniz rule

$$D^\ell_\alpha(h_1 \cdots h_k) = \sum_{j_1+\cdots+j_k=\ell} \binom{\ell}{j_1, \ldots, j_k} D^{j_1}_\alpha(h_1) \cdot \cdots \cdot D^{j_k}_\alpha(h_k).$$

If $j_1 + \cdots + j_k = \ell$, and $k > \ell$, then at least one of the j_i is 0 (by the pigeonhole principle). Thus, when $k > \ell$, for a typical generator $g = h_1 \cdots h_k$ of $\mathfrak{m}^k_p$, we have that

$$\binom{\ell}{j_1, \ldots, j_k} D^{j_1}_\alpha(h_1) \cdot \cdots \cdot D^{j_k}_\alpha(h_k) \text{ is an element of } \mathfrak{m}_p \text{ for all } j_1, \ldots, j_k \text{ with } j_1 + \cdots + j_k = \ell,$$

and hence $D^\ell_\alpha(g) \in \mathfrak{m}_p$, being a sum of terms of the above form (equivalently, $D^\ell_\alpha(g)(p) = 0$).

By linearity of D^ℓ_α, it follows that $D^\ell_\alpha(\mathfrak{m}^k_p) \subset \mathfrak{m}_p$ whenever $k > \ell$. In particular, we have:

(2.12) $\qquad$ If $g \in \mathfrak{m}^{r+1}_p$, $\qquad$ then $D^0_\alpha(g)(p) = 0,\ D^1_\alpha(g)(p) = 0,\ \ldots,\ D^r_\alpha(g)(p) = 0.$

Let $\hat{\mathcal{O}}_{X,p}$ denote the completion of $\mathcal{O}_{X,p}$ at the maximal ideal $\mathfrak{m}_p$, and $\hat{\mathfrak{m}}_p$ denote the image of $\mathfrak{m}_p$ in $\hat{\mathcal{O}}_{X,p}$. By the Cohen structure theorem (as a reminder, the preceding reductions brought us to the case when $\mathcal{O}_{X,p}$ is a regular local ring),

$$\hat{\mathcal{O}}_{X,p} \cong k[\![x_1, \ldots, x_n]\!],$$

where $n = \dim_k \mathfrak{m}_p/\mathfrak{m}_p^2$, and $\hat{\mathfrak{m}}_p$ goes to the ideal $(x_1, \ldots, x_n)$ under the above identification. Moreover, we have

$$\mathfrak{m}_p^k/\mathfrak{m}_p^{r+1} \cong \hat{\mathfrak{m}}_p^k/\hat{\mathfrak{m}}_p^{r+1} \cong (x_1, \ldots, x_n)^k/(x_1, \ldots, x_n)^{r+1} \qquad \text{for all } 0 \le k \le r+1.^5 \tag{2.13}$$

By observation (2.12), if $\tilde{g},\ \tilde{g}' \in \mathfrak{m}_p^k$ are representatives of $g \in \mathfrak{m}_p^k/\mathfrak{m}_p^{r+1}$, we have

$$D_\alpha^\ell(\tilde{g})(p) = D_\alpha^\ell(\tilde{g}')(p) \qquad \text{for all } 0 \le \ell \le r.$$

Therefore, it is enough to find $f_1, \ldots, f_{r+1} \in \mathcal{O}_{X,p}/\mathfrak{m}_p^{r+1}$ so that the determinant

$$\det \begin{pmatrix} f_1 & \cdots & f_{r+1} \\ D_\alpha^1(f_1) & \cdots & D_\alpha^1(f_{r+1}) \\ \vdots & \ddots & \vdots \\ D_\alpha^r(f_1) & \cdots & D_\alpha^r(f_{r+1}) \end{pmatrix} (p) \tag{2.14}$$

does not vanish — choosing representatives $\tilde{f}_j$ in $\mathcal{O}_{X,p}$, it would follow that $\det\big(D_\alpha^\ell(\tilde{f}_k)\big)$ does not vanish at p, hence does not vanish on a neighbourhood U of p; by the last reduction step, this would finish the proof of the lemma.

Therefore, for the remainder of the proof, we work modulo $\mathfrak{m}_p^{r+1}$, and make the identifications of (2.13).

Define $h_k \in \mathcal{O}_X(U_\alpha)$ by $h_k\, s_\alpha = D(U_\alpha)(x_k)$, $\quad 1 \le k \le n$ (recall that $D = \phi \circ d$). Since ϕ is surjective, there is at least one $j \in \{1, \ldots, n\}$ with $h_k(p) \neq 0$. Fix such a j.

Since X is smooth, for any $f \in \mathcal{O}_X(U_\alpha)$, we have

$$D(U_\alpha)(f) = \left(\frac{\partial f}{\partial x_1}\, h_1 + \cdots + \frac{\partial f}{\partial x_n}\, h_n \right) s_\alpha$$

[5] By convention, $(x_1, \ldots, x_n)^0 = k[x_1, \ldots, x_n]$.

Therefore, for $\ell \geq 1$,

$$D^{\ell}_{\alpha}(f) = \sum_{|\alpha|=\ell} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} h_1^{\alpha_1} \cdots h_n^{\alpha_n} \tag{2.15}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Set

$$f_1 = 1, \quad f_2 = x_j, \quad \ldots, \quad f_k = x_j^{k-1}, \quad \ldots, \quad f_{r+1} = x_j^r.$$

We first verify that the matrix of (2.14) has zeroes above the diagonal — indeed, evidently $f_k \in \mathfrak{m}_p^{k-1}$, so that by observation (2.12), $D^{\ell}_{\alpha}(f_k) = 0$ for $0 \leq \ell \leq k-2$. These are exactly the terms that are strictly above the diagonal.

On the other hand, the kth diagonal term is

$$D^{k-1}_{\alpha}(f_k) = D^{k-1}_{\alpha}(x_j^{k-1}) = (k-1)!\, h_j^{k-1} \neq 0,$$

by the choice of j.

Therefore, for the collection $\{f_k = x_j^{k-1}\}$, the matrix of (2.14) is lower-triangular with nonzero terms along the diagonal, and hence has nonzero determinant. The proof is complete.

□

In principle, Lemma 2.14 yields a method for recursively computing the transition functions of $\mathcal{P}^r_{\phi}$ up to arbitrary r. One proceeds as follows:

Let $s_{\alpha} : U_{\alpha} \to M$ be a collection of local frames, with transition functions $s_{\beta} = \tau_{\beta\alpha} s_{\alpha}$. For $f \in U_{\alpha\beta}$ (omitting restriction symbols to simplify notation),

$$D(f) = D^1_{\alpha}(f) s_{\alpha} = D^1_{\beta}(f) s_{\beta} = D^1_{\beta}(f) \tau_{\beta\alpha} s_{\alpha},$$

so that

$$D^1_{\beta}(f) = \tau_{\alpha\beta} D^1_{\alpha}(f).$$

Then,

$$
\begin{aligned}
D^2_\beta(f) &= D^1_\beta(D^1_\beta(f)) \\
&= D^1_\beta(\tau_{\alpha\beta} D^1_\alpha(f)) \\
&= \tau_{\alpha\beta} D^1_\alpha(\tau_{\alpha\beta} D^1_\alpha(f)) \\
&= \tau_{\alpha\beta}\left(D^1_\alpha(\tau_{\alpha\beta}) D^1_\alpha(f) + \tau_{\alpha\beta} D^2_\alpha(f)\right) \\
&= \left(\tau_{\alpha\beta} D^1_\alpha(\tau_{\alpha\beta})\right) D^1_\alpha(f) + \tau^2_{\alpha\beta}\, D^2_\alpha(f).
\end{aligned}
$$

Continuing inductively, we obtain, for example:

Example 2.15. Suppose that M is trivialized over the cover $\{U_\alpha\}$, and let $\tau_{\alpha\beta}$ be the transition functions for M. With respect to the induced local frames on $\mathcal{P}^5_\phi$, the transition functions (acting on sections) are (for simplicity of notation, $\tau_{\alpha\beta}$ is shortened to τ)

$$
\tau'_{5,\alpha\beta} = \left(\begin{array}{c|l|l|l|c|c}
1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \tau & 0 & 0 & 0 & 0 \\ \hline
0 & \tau D^1_\alpha(\tau) & \tau^2 & 0 & 0 & 0 \\ \hline
0 & \tau D^1_\alpha(\tau)^2 + \tau^2 D^2_\alpha(\tau) & 3\,\tau^2 D^1_\alpha(\tau) & \tau^3 & 0 & 0 \\ \hline
0 & \begin{array}{l} \tau D^1_\alpha(\tau)^3 \quad + \\ 4\,\tau^2 D^1_\alpha(\tau) D^2_\alpha(\tau) \quad + \\ \tau^3 D^3_\alpha(\tau) \end{array} & 7\,\tau^2 D^1_\alpha(\tau)^2 + 4\,\tau^3 D^2_\alpha(\tau) & 6\,\tau^3 D^1_\alpha(\tau) & \tau^4 & 0 \\ \hline
0 & \begin{array}{l} \tau D^1_\alpha(\tau)^4 \quad + \\ 11\,\tau^2 D^1_\alpha(\tau)^2 D^2_\alpha(\tau) \quad + \\ 4\,\tau^3 D^2_\alpha(\tau)^2 \quad + \\ 7\,\tau^3 D^1_\alpha(\tau) D^3_\alpha(\tau) \quad + \\ \tau^4 D^4_\alpha(\tau) \end{array} & \begin{array}{l} 15\,\tau^2 D^1_\alpha(\tau)^3 \quad + \\ \tau^3 D^1_\alpha(\tau) D^2_\alpha(\tau) \quad + \\ 5\,\tau^4 D^3_\alpha(\tau) \end{array} & \begin{array}{l} 25\,\tau^3 D^1_\alpha(\tau)^2 \quad + \\ 10\,\tau^4 D^2_\alpha(\tau) \end{array} & 10\,\tau^4 D^1_\alpha(\tau) & \tau^5
\end{array}\right)
$$

For each $0 \le i \le 5$, the transition function $\tau'_{i,\alpha\beta}$ of $\mathcal{P}^i_\phi$ appears as the top-left $(i \times i)$ minor of $\tau'_{5,\alpha\beta}$.

2.4 Functoriality

Let $\pi : Y \to X$ be a morphism of schemes, M be a line bundle on X, $\phi_X : \Omega_X \to M$ be an $\mathcal{O}_X$-linear map, and $\phi_Y : \Omega_Y \to \pi^* M$ be an $\mathcal{O}_Y$-linear map. Suppose that ϕ_X and ϕ_Y satisfy the equality

$$\pi^* \phi_X = \phi_Y \circ d\pi. \tag{2.16}$$

(The diagram

$$\begin{tikzcd}
\mathcal{O}_Y \arrow[r, "d_Y"] & \Omega_Y \arrow[r, "\phi_Y"] & \pi^* M \\
\pi^* \mathcal{O}_X \arrow[u, equal, "\pi^*"] \arrow[r, "\pi^* d_X"'] & \pi^* \Omega_X \arrow[u, "d\pi"] \arrow[ur, "\pi^* \phi_X"'] &
\end{tikzcd}$$

then commutes, so that the similar equality $\pi^* D_X = D_Y \circ \pi^*$ between derivations holds.)

The goal of this section, achieved in Theorem 2.24, is, given the above hypotheses, to construct an isomorphism of $\mathcal{O}_Y$-algebras

$$\gamma_{\mathrm{I}}^r(\mathcal{F}) : \pi_{\mathrm{I}}^* \left(\mathcal{P}_{\phi_X}^r(\mathcal{F})\right) \to \mathcal{P}_{\phi_Y}^r(\pi^* \mathcal{F}),$$

where π_{I}^* denotes pullback with respect to the ψ_{I}-induced $\mathcal{O}_X$-module structure, and $\mathcal{P}_{\phi_Y}^r(\pi^* \mathcal{F})$ is taken with its ψ_{I}-induced $\mathcal{O}_Y$-algebra structure.

Moreover, the isomorphisms $\gamma_{\mathrm{I}}^r(\mathcal{F})$ should have the property that for $V \subset H^0(X, \mathcal{F})$, and $\pi^* V$ the pulled-back linear series, the diagram

$$\begin{tikzcd}
\pi^* (V \otimes_k \mathcal{O}_X) \arrow[r] \arrow[d] & \pi_{\mathrm{I}}^* \left(\mathcal{P}_{\phi_X}^r(\mathcal{F})\right) \arrow[d, "\gamma_{\mathrm{I}}^r(\mathcal{F})"', "\cong"] \\
\pi^* V \otimes_k \mathcal{O}_Y \arrow[r] & \mathcal{P}_{\phi_Y}^r(\pi^* \mathcal{F})
\end{tikzcd}$$

commutes.

Outline of the proof. The arguments of this section are simple but lengthy, and for the reader's convenience we include here an outline of the various pieces of the proof.

The proof begins by constructing morphisms of sheaves of rings (not yet $\pi^{-1}\mathcal{O}_X$-algebras)

$$\gamma^r : \pi^{-1}\mathcal{P}^r_{\phi_X} \to \mathcal{P}^r_{\phi_Y}$$

that are compatible with π^r, ψ^r_{I} and ψ^r_{II}. The morphisms γ^r are constructed just after the proof of Lemma 2.17. Lemmas 2.16 and 2.17 provide general results that are used for the construction of γ^r; Proposition 2.18 checks compatibility of γ^r with projection maps, Proposition 2.19 checks that γ^r are morphisms of sheaves of rings, and Proposition 2.20 checks that γ^r are compatible with ψ^r_{I} and ψ^r_{II}.

Once the morphisms sheaves of rings γ^r have been constructed, we modify them to obtain morphisms *of "$\mathcal{O}_Y$-modules"* $\pi^*\mathcal{P}^r_{\phi_X} \to \mathcal{P}^r_{\phi_Y}$. Because both the source and target have more than one $\mathcal{O}_Y$-module structure, we must be careful to specify which module structures we mean.

We now describe the modification of γ^r when both source and target have ψ_{I}-induced structures. We have the commuting diagram of $\pi^{-1}\mathcal{O}_X$-algebras

$$\begin{tikzcd}
\pi^{-1}\mathcal{P}^r_{\phi_X} \arrow[r, "\gamma^r"] & \mathcal{P}^r_{\phi_Y} \\
\pi^{-1}\mathcal{O}_X \arrow[u, "\pi^{-1}\psi^r_{\mathrm{I}}"] \arrow[r, "\pi^{\#}"] & \mathcal{O}_Y \arrow[u, "\psi^r_{\mathrm{I}}"']
\end{tikzcd}$$

where $\pi^{-1}\mathcal{P}^r_{\phi_X}$ has its $\pi^{-1}\psi^r_{\mathrm{I}}$-induced structure and $\mathcal{P}^r_{\phi_Y}$ has its ψ^r_{I}-induced structure. Using the fact that the tensor product of two $\pi^{-1}\mathcal{O}_X$-algebras satisfies the universal property of the pushout, we obtain the dashed morphism below:

$$\begin{tikzcd}
\pi^{-1}\mathcal{P}^r_{\phi_X} \arrow[rr, "\gamma^r"] \arrow[dr] & & \mathcal{P}^r_{\phi_Y} \\
& \pi^{-1}\mathcal{P}^r_{\phi_X} {}_{\mathrm{I}}\otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_Y \arrow[ur, dashed, "\exists"] & \\
\pi^{-1}\mathcal{O}_X \arrow[uu, "\pi^{-1}\psi^r_{\mathrm{I}}"] \arrow[rr, "\pi^{\#}"] & & \mathcal{O}_Y \arrow[uu, "\psi^r_{\mathrm{I}}"'] \arrow[ul]
\end{tikzcd}$$

(The above description ignores a technicality, which is taken care of by Lemma 2.21.)

This dashed morphism is exactly the desired modification of γ^r; it is a morphism of $\mathcal{O}_Y$-modules with ψ_{I}-induced module structures on source and target. We denote the dashed morphism by γ^r_{I}. In fact, γ^r_{I} is an *isomorphism*, which can be proven by first proving that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi^*(M^{\otimes r}) \arrow[r] \arrow[d] & \pi^{-1}\mathcal{P}^r_{\phi_X} {}_{\mathrm{I}}\otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_Y \arrow[r] \arrow[d] & \pi^{-1}\mathcal{P}^{r-1}_{\phi_X} {}_{\mathrm{I}}\otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\pi^* M)^{\otimes r} \arrow[r] & \mathcal{P}^r_{\phi_Y} \arrow[r] & \mathcal{P}^{r-1}_{\phi_Y} \arrow[r] & 0
\end{tikzcd}$$

commutes in Corollary 2.22, and then using induction on r and the five-lemma. The isomorphism γ^r_{II} can be constructed entirely analogously.

Finally, in the last step of the argument just prior to the statement of Theorem 2.24, we check that γ^r_{I} can be further extended to an isomorphism of $\mathcal{O}_Y$-modules

$$\gamma^r_{\mathrm{I}}(\mathcal{F}) \colon \pi^*_{\mathrm{I}}\left(\mathcal{P}^r_{\phi_X}(\mathcal{F})\right) \to \mathcal{P}^r_{\phi_Y}(\pi^*\mathcal{F}),$$

where π^*_{I} denotes pullback with respect to the ψ_{I}-induced $\mathcal{O}_X$-module structure, and $\mathcal{P}^r_{\phi_Y}(\pi^*\mathcal{F})$ is taken with its ψ_{I}-induced $\mathcal{O}_Y$-module structure. In the proof of Theorem 2.24 proper, it only remains to check that $\gamma^r_{\mathrm{I}}(\mathcal{F})$ is well-behaved with respect to global sections.

The Proof. Let $\pi : Y \to X$ be a morphism of topological spaces, and let $\mathcal{F}$ be a sheaf of abelian groups on X. We remind the reader that for any point Y of y, the stalks $(\pi^{-1}\mathcal{F})_y$ and $\mathcal{F}_{\pi(y)}$ may be canonically identified, as follows. Write $x = \pi(y)$, and let $i_y : \{y\} \to Y$ and $i_x : \{x\} \to X$ be the inclusion maps. Let $\mathcal{F}$ be a sheaf (of abelian groups, say) on X. By the commutativity of the diagram

$$\begin{tikzcd} \{y\} \arrow[r, "i_y"] \arrow[d, "\pi|_{\{y\}}"'] & Y \arrow[d, "\pi"] \\ \{x\} \arrow[r, "i_x"] & X \end{tikzcd}$$

and functoriality of inverse image (denoting constant sheaves over a point by the corresponding abelian group, and canonical isomorphisms by equalities),

$$\begin{aligned} \left(\pi^{-1}\mathcal{F}\right)_y &= i_y^{-1}\left(\pi^{-1}\mathcal{F}\right) \\ &= \pi|_{\{y\}}^{-1}\left(\mathcal{F}_x\right) \\ &= \mathcal{F}_x. \end{aligned}$$

For any open subset U of X, we have the map

$$\mathcal{F}(U) \to (\pi^{-1}\mathcal{F})(\pi^{-1}(U)), \tag{2.17}$$

obtained as follows. By definition, $\pi^{-1}\mathcal{F}$ is the sheafification of the presheaf that assigns to $W \subset Y$ the direct limit $\operatorname{colim}_{U \supset \pi(W)} \mathcal{F}(U)$. Since $U \supset \pi(\pi^{-1}(U))$, there is a map $\mathcal{F}(U) \to \operatorname{colim}_{U \supset \pi(\pi^{-1}(U))} \mathcal{F}(U)$, which can be composed with the sheafification map to obtain the desired map $\mathcal{F}(U) \to (\pi^{-1}\mathcal{F})(\pi^{-1}(U))$. Let $f \in \mathcal{F}(U)$ and denote the image of f in $\pi^{-1}\mathcal{F}(\pi^{-1}(U))$ under this map by g. Then for every $y \in \pi^{-1}(U)$, $g_y = f_{\pi(y)}$ under the identification $(\pi^{-1}\mathcal{F})_y = \mathcal{F}_{\pi(y)}$.

We begin with the following lemma on gluing together morphisms of sheaves defined over a base.

Lemma 2.16. *Let X and Y be topological spaces, $\pi : Y \to X$ a continuous map, and $\mathcal{B}_X$*

and $\mathcal{B}_Y$ bases for the topologies on X and Y, respectively. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X and Y, respectively.

For each pair of base open sets $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ with $V \subset \pi^{-1}(U)$, let γ_{UV} be a map of abelian groups

$$\gamma_{UV} : \mathcal{F}(U) \to \mathcal{G}(V),$$

and suppose that the collection $\{\gamma_{UV}\}_{U \in \mathcal{B}_X,\, V \in \mathcal{B}_Y,\, V \subset \pi^{-1}(U)}$ is compatible with restrictions, in the sense that for any inclusion of open subsets $U' \subset U$ of X and any inclusion of open subsets $V' \subset V$ of Y with $V \subset \pi^{-1}(U)$ and $V' \subset \pi^{-1}(U')$, the square

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\gamma_{UV}"] \arrow[d, "\mathrm{restr}_{U,U'}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{restr}_{V,V'}"] \\
\mathcal{F}(U') \arrow[r, "\gamma_{U'V'}"] & \mathcal{G}(V')
\end{tikzcd}$$

commutes.

(i) *For each point $y \in Y$, there is a unique induced map*

$$\tilde{\gamma}_y : \mathcal{F}_{\pi(y)} \to \mathcal{G}_y \tag{2.18}$$

of stalks satisfying the property that for each pair $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$ with $y \in V \subset \pi^{-1}(U)$ the square

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\gamma_{UV}"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\
\mathcal{F}_{\pi(y)} \arrow[r, "\tilde{\gamma}_y"] & \mathcal{G}_y
\end{tikzcd} \tag{2.19}$$

commutes.

(ii) *There is a unique morphism of sheaves*

$$\gamma : \pi^{-1}\mathcal{F} \to \mathcal{G}$$

satisfying either (hence both) of the following equivalent properties.

(a) *For each point $y \in Y$, the morphism $\gamma_y\colon (\pi^{-1}\mathcal{F})_y \to \mathcal{G}_y$ induced by γ is equal to $\tilde{\gamma}_y$ of (2.18) (under the identification of $(\pi^{-1}\mathcal{F})_y$ with $\mathcal{F}_{\pi(y)}$). Equivalently (by uniqueness of $\tilde{\gamma}_y$) for every base open set U of X and every base open set $y \in V \subset \pi^{-1}(U)$, the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\gamma_{UV}"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\ \mathcal{F}_{\pi(y)} \arrow[r, "\gamma_y"] & \mathcal{G}_y \end{tikzcd} \tag{2.20}$$

(b) *For each pair $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$, with $V \subset \pi^{-1}(U)$, the composition*

$$\mathcal{F}(U) \longrightarrow \pi^{-1}\mathcal{F}(\pi^{-1}(U)) \xrightarrow{\gamma(\pi^{-1}(U))} \mathcal{G}(\pi^{-1}(U)) \xrightarrow{\mathrm{restr}_{\pi^{-1}(U),V}} \mathcal{G}(V)$$

where the first map is the map to a direct limit followed by the sheafification map, is equal to γ_{UV}.

Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves of rings, and each γ_{UV} is a ring morphism, then γ is a morphism of sheaves of rings.

Proof. We begin by showing that property (ii)(a) (in the sense that the square (2.20) commutes) is equivalent to (ii)(b).

Suppose that property (ii)(a) holds for $\gamma\colon \pi^{-1}\mathcal{F} \to \mathcal{G}$. Let U be a base open subset of X and let $V \subset \pi^{-1}(U)$ be a base open subset of Y. We would like to show that the composition

$$\mathcal{F}(U) \to (\pi^{-1}\mathcal{F})(\pi^{-1}(U)) \to (\pi^{-1}\mathcal{F})(V) \to \mathcal{G}(V)$$

is equal to γ_{UV}.

Let $f \in \mathcal{F}(U)$ and denote the image of f in $(\pi^{-1}\mathcal{F})(\pi^{-1}(U))$ by h. Then for every $y \in V$,

$$(h|_V)_y = h_y = f_{\pi(y)},$$

so that

$$(\gamma(V)(h|_V))_y = \gamma_y((h|_V)_y) = \gamma_y\left(f_{\pi(y)}\right) = (\gamma_{UV}(f), V)$$

where the last equality holds by the commutativity of the square (2.20).

Since $\gamma(V)(h|_V)$ and $\gamma_{UV}(f)$ have equal germs at each $y \in V$, $\gamma(V)(h|_V) = \gamma_{UV}(f)$. The desired conclusion follows by construction of h.

Conversely, suppose that (ii)(b) holds for $\gamma\colon \pi^{-1}\mathcal{F} \to \mathcal{G}$. Fix $y \in Y$, U a base open subset of X and $y \in V \subset \pi^{-1}(U)$ a base open subset of Y. Let $f \in \mathcal{F}(U)$, and again denote the image of f in $(\pi^{-1}\mathcal{F})(\pi^{-1}(U))$ by h. We then have $h_y = f_{\pi(y)}$, and by hypothesis $\gamma(V)(h|_V) = \gamma_{UV}(f)$. But this implies that

$$(\gamma_{UV}(f))_y = (\gamma(V)(h|_V))_y = \gamma_y((h|_V)_y) = \gamma_y(f_{\pi(y)}),$$

showing that the square (2.20) commutes.

We proceed to prove part (i), namely that for each point $y \in Y$ there is an induced morphism of stalks $\mathcal{F}_{\pi(y)} \to \mathcal{G}_y$. Let $y \in Y$. We make use of the universal property of the direct limit $\mathcal{F}_{\pi(y)} = \operatorname{colim}_{\pi(y)\in W} \mathcal{F}(W)$.

Let $\pi(y) \in W$ be an open subset of X. There is a base open set $\pi(y) \in U \subset W$. We obtain a map

$$\mathcal{F}(W) \xrightarrow{\mathrm{restr}_{W,U}} \mathcal{F}(U) \xrightarrow{\gamma_{U,\pi^{-1}(U)}} \mathcal{G}(\pi^{-1}(U)) \longrightarrow \mathcal{G}_y\ .$$

We check that the map $\mathcal{F}(W) \to \mathcal{G}_y$ does not depend on the choice of U. Let U' be another base open set with $\pi(y) \in U' \subset W$. There is a base open set $\pi(y) \in V \subset U \cap U'$, and the diagram

$$\begin{tikzcd}
 & \mathcal{F}(U) \arrow[r, "\gamma_{U,\pi^{-1}(U)}"] \arrow[d, "\mathrm{restr}_{U,V}"'] & \mathcal{G}(\pi^{-1}(U)) \arrow[d, "\mathrm{restr}_{\pi^{-1}(U),\pi^{-1}(V)}"] \arrow[dr, bend left] & \\
\mathcal{F}(W) \arrow[ur, bend left] \arrow[r] \arrow[dr, bend right] & \mathcal{F}(V) \arrow[r, "\gamma_{V,\pi^{-1}(V)}"] & \mathcal{G}(\pi^{-1}(V)) \arrow[r] & \mathcal{G}_y \\
 & \mathcal{F}(U') \arrow[r, "\gamma_{U',\pi^{-1}(U')}"] \arrow[u, "\mathrm{restr}_{U',V}"] & \mathcal{G}(\pi^{-1}(U')) \arrow[u, "\mathrm{restr}_{\pi^{-1}(U'),\pi^{-1}(V)}"'] \arrow[ur, bend right] &
\end{tikzcd}$$

commutes (the two middle squares commute by compatibility of (γ_{UV}) with restriction maps). It follows that the map $\mathcal{F}(W) \to \mathcal{G}_y$ does not depend on the choice of U. We denote this map by $\tilde{\gamma}(W)$.

Now suppose that W' is another open subset of X with $\pi(y) \in W' \subset W$. As there is a base open set U with $\pi(y) \in U \subset W' \subset W$, and the diagram

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[d] \arrow[dr] & & & \\
\mathcal{F}(W') \arrow[r] & \mathcal{F}(U) \arrow[r, "\gamma_{U,\pi^{-1}(U)}"] & \mathcal{G}(\pi^{-1}(U)) \arrow[r] & \mathcal{G}_y
\end{tikzcd}$$

commutes, it follows that the diagram

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[dd, "\mathrm{restr}_{W,W'}"'] \arrow[dr, "\tilde{\gamma}(W)"] & \\
 & \mathcal{G}_y \\
\mathcal{F}(W') \arrow[ur, "\tilde{\gamma}(W')"'] &
\end{tikzcd}$$

commutes. As this is true for any inclusion $\pi(y) \in W' \subset W$, by the universal property of

$\mathcal{F}_{\pi(y)}$, there is an induced map $\tilde{\gamma}_y : \mathcal{F}_{\pi(y)} \to \mathcal{G}_y$ making the diagram

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[dd] \arrow[dr] \arrow[drr, bend left, "\tilde{\gamma}(W)"] & & \\
& \mathcal{F}_{\pi(y)} \arrow[r, "\tilde{\gamma}_y"] & \mathcal{G}_y \\
\mathcal{F}(W') \arrow[ur] \arrow[urr, bend right, "\tilde{\gamma}(W')"'] & &
\end{tikzcd}$$

commute.

Now suppose that U is a base open set of X containing $\pi(y)$ and $V \subset \pi^{-1}(U)$ is a base open set of Y. The inner square (drawn as a trapezoid) of the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\gamma_{U,\pi^{-1}(U)}"] \arrow[d] & \mathcal{G}(\pi^{-1}(U)) \arrow[r] \arrow[dr] & \mathcal{G}(V) \arrow[d] \\
\mathcal{F}_{\pi(y)} \arrow[rr, "\tilde{\gamma}_y"] & & \mathcal{G}_y
\end{tikzcd}$$

commutes by construction of $\tilde{\gamma}_y$, and the triangle on the upper-right commutes for any restriction, hence the outer square of the diagram, which is the square (2.19) in the statement of part (i), also commutes.

Finally, suppose that $y \in Y$ and that $\tilde{\psi}_y : \mathcal{F}_{\pi(y)} \to \mathcal{G}_y$ is a map of stalks with the property that

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\gamma_{UV}"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\
\mathcal{F}_{\pi(y)} \arrow[r, "\tilde{\psi}_y"] & \mathcal{G}_y
\end{tikzcd} \tag{2.21}$$

commutes for each pair $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$ with $y \in V \subset \pi^{-1}(U)$. Let (f, U) be a representative of a germ of $\mathcal{F}_{\pi(y)}$, where after possibly shrinking U may be assumed to be a base open set, and $f \in \mathcal{F}(U)$. Let V be a base open set with $y \in V \subset \pi^{-1}(U)$. Then by square (2.21) above and the similar square (2.19) for $\tilde{\gamma}_y$,

$$\tilde{\gamma}_y\left((f,U)\right) = (\gamma_{UV}(f), V) = \tilde{\psi}_y\left((f,U)\right),$$

so that $\tilde{\gamma}_y = \tilde{\psi}_y$, showing that $\tilde{\gamma}_y$ is unique.

The proof of part (i) is complete, and we proceed to the proof of part (ii), beginning with the construction of a morphism $\gamma\colon \pi^{-1}\mathcal{F} \to \mathcal{G}$. It is useful to introduce the following notion for the construction of γ:

Let $W \subset Y$ be an open set and let $f \in (\pi^{-1}\mathcal{F})(W)$. A *base decomposition* $\mathscr{D}(f,W)$ *of* f *over* W is a collection $(U_i,\, V_i,\, f_i)_{i\in\mathcal{I}}$, where U_i are base open sets of X and V_i are base open sets of Y, with $V_i \subset \pi^{-1}(U_i)$ for each i and $\bigcup_i V_i = W$, and where f_i is a section in $\mathcal{F}(U_i)$ such that for each point y of V_i we have $(f_i)_{\pi(y)} = f_y$ (under the identification of $\mathcal{F}_{\pi(y)}$ with $(\pi^{-1}\mathcal{F})_y$).

- *For any* $W \subset Y$ *open and any* $f \in (\pi^{-1}\mathcal{F})(W)$*, there is a base decomposition of* f *over* W. Fix $y \in W$. Then f determines a germ f_y of $(\pi^{-1}\mathcal{F})_y$. Choose a representative $(g \in \mathcal{F}(U), U)$ of the germ that corresponds to f_y under the identification $(\pi^{-1}\mathcal{F})_y = \mathcal{F}_{\pi(y)}$ (we have $\pi(y) \in U \subset X$). After possibly shrinking, we may assume that U is a base open subset containing $\pi(y)$. Let g' denote the image of g under the map $\mathcal{F}(U) \to \pi^{-1}\mathcal{F}(\pi^{-1}(U))$; then $(g')_y = g_{\pi(y)} = f_y$. Since $(g')_y = f_y$, there is an open set $V \subset \pi^{-1}(U) \cap W$ (containing y) with $g'|_V = f|_V$. After possibly shrinking V, we may assume that V is a base open subset of Y. We obtain a triple $U \subset X,\ y \in V \subset \pi^{-1}(U)$ (with U, V base open sets in their respective spaces) and $g \in \mathcal{F}(U)$ satisfying $g_{\pi(y')} = f_{y'}$ for all $y' \in V$. Since y was an arbitrary point of W, this establishes the existence of a base decomposition of f over W.

- *Given a base decomposition* $\mathscr{D}$ *of* f *over* W*, there is a corresponding section* $\gamma(W)(f)_{\mathscr{D}}$ *of* $\mathcal{G}(W)$. Given $\mathscr{D} = (U_i,\, V_i,\, f_i)_{i\in\mathcal{I}}$, define $g_i := \gamma_{U_i,V_i}(f_i) \in \mathcal{G}(V_i)$. For each $y \in V_i \cap V_j =:$

V_{ij},

$$(g_i)_y = \tilde{\gamma}_y((f_i)_{\pi(y)}) = \tilde{\gamma}_y(f_{\pi(y)}) = \tilde{\gamma}_y((f_j)_{\pi(y)}) = (g_j)_y,$$

so that g_i and g_j have the same germ for each point of V_{ij}. It follows that $g_i|_{V_{ij}} = g_j|_{V_{ij}}$, and so the sections g_i glue together to a unique section $g \in \mathcal{G}(W)$ with $g|_{V_i} = g_i$ for each i. Denote the section g constructed in this way by $\gamma(W)(f)_{\mathscr{D}}$.

Note that for each $y \in W$, by construction

$$(\gamma(W)(f)_{\mathscr{D}})_y = \tilde{\gamma}_y(f_{\pi(y)}).$$

The right side of the above equality is independent of $\mathscr{D}$.

- *For any two base decompositions $\mathscr{D}$, $\mathscr{D}'$ of f over W, $\gamma(W)(f)_{\mathscr{D}} = \gamma(W)(f)_{\mathscr{D}'}$.* Denote $\gamma(W)(f)_{\mathscr{D}}$ by g and $\gamma(W)(f)_{\mathscr{D}'}$ by g'. For each $y \in W$,

$$g_y = \tilde{\gamma}_y(f_{\pi(y)}) = g'_y.$$

Since g and g' have the same germ at each point y of W, $g = g'$.

Denote by $\gamma(W)(f)$ the section $\gamma(W)(f)_{\mathscr{D}}$, where $\mathscr{D}$ is any base decomposition of f over W. The preceding paragraph shows that $\gamma(W)(f)$ is well-defined.

- *For any two sections f, f' in $(\pi^{-1}\mathcal{F})(W)$, $\gamma(W)(f+f') = \gamma(W)(f)+\gamma(W)(f')$.* Denote the sections $\gamma(W)(f)$, $\gamma(W)(f')$ and $\gamma(W)(f+f')$ by g, g' and g'', respectively. For each $y \in W$,

$$(g+g')_y = g_y + g'_y = \tilde{\gamma}_y(f_{\pi(y)}) + \tilde{\gamma}_y(f'_{\pi(y)}) = \tilde{\gamma}_y\left(f_{\pi(y)} + f'_{\pi(y)}\right) = \tilde{\gamma}_y\left((f+f')_{\pi(y)}\right) = (g'')_y.$$

Since $g+g'$ and g'' have the same germ at each point of W, we have $g+g' = g''$, as desired.

- *For any $W' \subset W \subset X$, and any $f \in (\pi^{-1}\mathcal{F})(W)$, $\gamma(W)(f)|_{W'} = \gamma(W')(f|_{W'})$.* For each $y \in W'$, we have

$$\left(\gamma(W)(f)|_{W'}\right)_y = \tilde{\gamma}_y(f_{\pi(y)}) = \left(\gamma(W')(f|_{W'})\right)_y,$$

so that once again, the sections $\gamma(W)(f)|_{W'}$ and $\gamma(W')(f|_{W'})$ are equal, having equal germs at each point of W'.

- *Suppose that $\mathcal{F}, \mathcal{G}$ are sheaves of rings, and each γ_{UV} is a morphism of sheaves of rings. For any two sections f, f' in $(\pi^{-1}\mathcal{F})(W)$, $\gamma(W)(ff') = \gamma(W)(f)\gamma(W)(f')$.* In this case, the morphism $\tilde{\gamma}_y$ is a ring morphism (this is immediate from the category-theoretic construction of $\tilde{\gamma}_y$). The statement is then proven similarly to preservation of addition: denote the sections $\gamma(W)(f)$, $\gamma(W)(f')$ and $\gamma(W)(ff')$ by g, g' and g'', respectively. For each $y \in W$,

$$(gg')_y = (g_y)(g'_y) = \tilde{\gamma}_y(f_{\pi(y)})\tilde{\gamma}_y(f'_{\pi(y)}) = \tilde{\gamma}_y\left(f_{\pi(y)}f'_{\pi(y)}\right) = \tilde{\gamma}_y\left((ff')_{\pi(y)}\right) = (g'')_y.$$

Since gg' and g'' have the same germ at each point of W, we have $gg' = g''$, as desired.

The above steps complete the construction of a morphism $\gamma\colon \pi^{-1}\mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups (and verification that it is indeed a morphism). We verify that γ satisfies (ii)(a), that is, that for each $y \in Y$, the induced morphism $\gamma_y\colon (\pi^{-1}\mathcal{F})_y \to \mathcal{G}_y$ on stalks is equal to $\tilde{\gamma}_y$. Fix $y \in Y$. Let (f, W) be a representative of a germ in $(\pi^{-1}\mathcal{F})_y$, with $f \in (\pi^{-1}\mathcal{F})(W)$, and denote by $f_{\pi(y)}$ the germ in $\mathcal{F}_{\pi(y)}$ identified with (f, W). By construction, the image of f under $\gamma(W)$ has germ

$$(\gamma(W)(f))_y = \tilde{\gamma}_y(f_{\pi(y)}) \qquad \text{at } y.$$

On the other hand, since γ is a sheaf map, $\gamma_y((f, W)) = (\gamma(W)(f))_y$, so that

$$\gamma_y((f, W)) = \tilde{\gamma}_y(f_{\pi(y)}).$$

Since the germ (f, W) and the point y were chosen arbitrarily, this shows that $\gamma_y = \tilde{\gamma}_y$ for all $y \in Y$ (after identifying $(\pi^{-1}\mathcal{F})_y$ with $\mathcal{F}_{\pi(y)}$).

Property (ii)(a) is verified for γ, and therefore (ii)(b) holds as well.

We prove the induced morphism γ is unique. By uniqueness of $\tilde{\gamma}_y$, if $\gamma, \gamma' : \pi^{-1}\mathcal{F} \to \mathcal{G}$ are morphisms that both satisfy property (ii)(a), then $\gamma_y = \tilde{\gamma}_y = \gamma'_y$ for each $y \in Y$, hence $\gamma = \gamma'$. □

Lemma 2.17. *Let X and Y be topological spaces, $\pi : Y \to X$ a continuous map, and $\mathcal{B}_X$ and $\mathcal{B}_Y$ bases for the topologies on X and Y, respectively.*

Let $\mathcal{F}$, $\mathcal{K}$ be sheaves of abelian groups on X, $\phi : \mathcal{F} \to \mathcal{K}$ a morphism of sheaves of abelian groups, and let $\mathcal{G}$, $\mathcal{L}$ be sheaves of abelian groups on Y, $\xi : \mathcal{G} \to \mathcal{L}$ a morphism of sheaves of abelian groups.

For each pair of base open sets $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ with $V \subset \pi^{-1}(U)$, let γ_{UV} and ψ_{UV} be maps of abelian groups

$$\gamma_{UV} : \mathcal{F}(U) \to \mathcal{G}(V) \qquad \text{and} \qquad \psi_{UV} : \mathcal{K}(U) \to \mathcal{L}(V),$$

and suppose that the collections $\{\gamma_{UV}\}_{U \in \mathcal{B}_X, V \in \mathcal{B}_Y, V \subset \pi^{-1}(U)}$ and $\{\psi_{UV}\}_{U \in \mathcal{B}_X, V \in \mathcal{B}_Y, V \subset \pi^{-1}(U)}$ are compatible with restrictions. Suppose, moreover, that for each pair of base open sets $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ with $V \subset \pi^{-1}(U)$, the following square is commutative:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\gamma_{UV}"] \arrow[d, "\phi(U)"'] & \mathcal{G}(V) \arrow[d, "\xi(V)"] \\ \mathcal{K}(U) \arrow[r, "\psi_{UV}"'] & \mathcal{L}(V) \end{tikzcd} \tag{2.22}$$

By Lemma 2.16, the collections $\{\gamma_{UV}\}$ and $\{\psi_{UV}\}$ induce unique morphisms $\gamma : \pi^{-1}\mathcal{F} \to \mathcal{G}$

and $\psi \colon \pi^{-1}\mathcal{K} \to \mathcal{L}$*, respectively. Then the square*

$$\begin{tikzcd} \pi^{-1}\mathcal{F} \arrow[r, "\gamma"] \arrow[d, "\pi^{-1}\phi"'] & \mathcal{G} \arrow[d, "\xi"] \\ \pi^{-1}\mathcal{K} \arrow[r, "\psi"'] & \mathcal{L} \end{tikzcd}$$

commutes.

Proof. It is sufficient to prove that $(\xi \circ \gamma)_y = (\psi \circ (\pi^{-1}\phi))_y$ for every $y \in Y$ (then $\xi \circ \gamma = \psi \circ (\pi^{-1}\phi)$ and the diagram commutes). Note that under the identifications $(\pi^{-1}\mathcal{F})_y = \mathcal{F}_{\pi(y)}$ and $(\pi^{-1}\mathcal{K})_y = \mathcal{K}_{\pi(y)}$, we have $(\pi^{-1}\phi)_y = \phi_{\pi(y)}$ for each $y \in Y$.

Let $y \in Y$, and choose a representative (f, U) of a germ in $\mathcal{F}_{\pi(y)}$. After possibly shrinking, U may be assumed to be a base open set of X. Let $V \subset \pi^{-1}(U)$ be a base open subset of Y containing y. We chase (f, U) through the left square:

$$\begin{tikzcd} \mathcal{F}_{\pi(y)} \arrow[r, "\gamma_y"] \arrow[d, "\phi_{\pi(y)}"'] & \mathcal{G}_y \arrow[d, "\xi_y"] \\ \mathcal{K}_{\pi(y)} \arrow[r, "\psi_y"'] & \mathcal{L}_y \end{tikzcd} \qquad \begin{tikzcd} (f, U) \arrow[r, mapsto] \arrow[dd, mapsto] & (\gamma_{UV}(f), V) \arrow[d, mapsto] \\ & (\xi(V)(\gamma_{UV}(f)), V) \arrow[d, equal] \\ (\phi(U)(f), U) \arrow[r, mapsto] & (\psi_{UV}(\phi(U)(f)), V) \end{tikzcd}$$

where the equality holds due to the commutativity of (2.22). Since (f, U) was the representative of an arbitrary germ, the statement is verified. $\square$

Remark. In above lemma, it is sufficient for ϕ and ξ to be defined over bases of X and Y, respectively. Then we obtain unique induced maps $\phi \colon \mathcal{F} \to \mathcal{K}$ and $\xi \colon \mathcal{G} \to \mathcal{L}$ and the same conclusion holds. As we will not require this statement, we will omit the proof.

Now, we specialize back to the set-up of the beginning of this section. Let $\pi \colon Y \to X$ be a morphism of schemes, M be a line bundle on X, $\phi_X \colon \Omega_X \to M$ be an $\mathcal{O}_X$-linear map, and $\phi_Y \colon \Omega_Y \to \pi^*M$ be an $\mathcal{O}_Y$-linear map. Suppose that ϕ_X and ϕ_Y satisfy equality (2.16): $\pi^*\phi_X = \phi_Y \circ d\pi$, so that also $\pi^* D_X = D_Y \circ \pi^*$, where $D_X := \phi_X \circ d_X$ and $D_Y = \phi_Y \circ d_Y$.

For each $r \geq 0$, we construct a morphism of sheaves of rings $\gamma^r : \pi^{-1}\mathcal{P}^r_{\phi_X} \to \mathcal{P}^r_{\phi_Y}$ that is compatible with projections, as well as the maps ψ_{I} and ψ_{II}, in a sense that is made precise below.

To begin, we construct γ^r as morphisms of sheaves of abelian groups.

We remind the reader that affine open sets are a basis for the topology of a scheme. Therefore, by Lemma 2.16, it is sufficient to construct for each affine open $\operatorname{Spec} A \subset X$ and each affine open $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A) \subset Y$ a morphism of abelian groups

$$\gamma^r_{AB} : \mathcal{P}^r_{\phi_X}(\operatorname{Spec} A) \to \mathcal{P}^r_{\phi_Y}(\operatorname{Spec} B)$$

compatible with restrictions.

Since $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$, we obtain a morphism $\pi|_{\operatorname{Spec} B} : \operatorname{Spec} B \to \operatorname{Spec} A$ of schemes. If $\operatorname{Spec} A' \subset \operatorname{Spec} A$ and $\operatorname{Spec} B' \subset \operatorname{Spec} B$ with $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ and $\operatorname{Spec} B' \subset \pi^{-1}(\operatorname{Spec} A')$, then the corresponding square

(2.23)
$$\begin{tikzcd} A \arrow[r, "\pi|^{\#}_{\operatorname{Spec} B}"] \arrow[d, "\mathrm{restr}"'] & B \arrow[d, "\mathrm{restr}"] \\ A' \arrow[r, "\pi|^{\#}_{\operatorname{Spec} B'}"'] & B' \end{tikzcd}$$

of pullbacks (and restrictions) is commutative.

From now on (unless noted otherwise), by abuse of notation, denote maps $\pi|^{\#}_{\operatorname{Spec} B} : A \to B$ by $\pi^{\#}$, omitting explicit reference to $\operatorname{Spec} B$.

Let $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ be affine open subsets. Denote $M(\operatorname{Spec} A)$ by M_A. Then $(\pi^* M)(\operatorname{Spec} B) = B \otimes_A M_A$. (We remind the reader that B is an A-module by $a \cdot b = \pi^{\#}(a)b$.) For each $r \geq 0$, denote $\mathcal{P}^r_{\phi_X}(\operatorname{Spec} A)$ by R^r_A, and $\mathcal{P}^r_{\phi_Y}(\operatorname{Spec} B)$ by S^r_B. For the latter, it will be useful to observe that

$$
\begin{aligned}
S_B^r = \mathcal{P}_{\phi_Y}^r(\operatorname{Spec} B) &= B \oplus \left(S_B^{r-1} \ {}_{\mathrm{II}}\!\otimes_B (B \otimes_A M_A)\right) \\
&\cong B \oplus \left(\left(S_B^{r-1} \ {}_{\mathrm{II}}\!\otimes_B B\right) \otimes_A M_A\right) \\
&\cong B \oplus \left(S_B^{r-1} \ {}_{\mathrm{II}}\!\otimes_A M_A\right).
\end{aligned}
$$

(Under the above two identifications, the element $(b, s \otimes (c \otimes m))$ is sent to $(b, (\psi_{\mathrm{II}}^r(c)s) \otimes m)$. It is quick to check that the projection map $\pi^r : S_B^r \to S_B^{r-1}$ and the product structure on S_B^r may be described as follows under the above isomorphisms: $\pi^r((b,\, s \otimes m)) = (b,\, \pi^{r-1}(s) \otimes m)$ and $(b,\, s \otimes m) \cdot (b',\, s' \otimes m') = (bb',\, \pi^r((b, s \otimes m))s' \otimes m' + \pi^r((b', s' \otimes m'))s \otimes m)$.)

Let $r \geq 0$ be an integer. We proceed to construct a collection of morphisms of abelian groups $\gamma_{AB}^r : \mathcal{P}_{\phi_X}^r(\operatorname{Spec} A) \to \mathcal{P}_{\phi_Y}^r(\operatorname{Spec} B)$ for each affine open $\operatorname{Spec} A \subset X$ and each affine open $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ that are compatible with restrictions. The construction proceeds by induction on r.

r = 0: Let $\operatorname{Spec} A' \subset \operatorname{Spec} A \subset X$ and $\operatorname{Spec} B' \subset \operatorname{Spec} B$ be affine open sets, with $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ and $\operatorname{Spec} B' \subset \pi^{-1}(\operatorname{Spec} A')$. Then

$$
\mathcal{O}_X(\operatorname{Spec} A) = A,\ \mathcal{O}_X(\operatorname{Spec} A') = A',\ \mathcal{O}_Y(\operatorname{Spec} B) = B,\ \mathcal{O}_Y(\operatorname{Spec} B') = B'.
$$

Set $\gamma_{AB}^0 = \pi|_{\operatorname{Spec} B}^{\#}$. As the square (2.23) is commutative, γ_{AB}^0 are compatible with restrictions.

r = 1: Let $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ be affine open sets. We have

$$
\begin{aligned}
\mathcal{P}_{\phi_X}^1(\operatorname{Spec} A) &= A \oplus M_A \qquad \text{and} \\
\mathcal{P}_{\phi_Y}^1(\operatorname{Spec} B) &= B \oplus (B \otimes_A M_A).
\end{aligned}
$$

Define

$$\gamma^1_{AB} : \mathcal{P}^1_{\phi_X}(\operatorname{Spec} A) \to \mathcal{P}^1_{\phi_Y}(\operatorname{Spec} B) \qquad \text{by} \qquad \gamma^1_{AB} : (a,\, m) \mapsto (\pi^{\#} a,\, 1 \otimes m).$$

The maps γ^1_{AB} are compatible with restrictions: let $\operatorname{Spec} A' \subset \operatorname{Spec} A$ and $\operatorname{Spec} B' \subset \operatorname{Spec} B$ with $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ and $\operatorname{Spec} B' \subset \pi^{-1}(\operatorname{Spec} A')$. Then for $(a, m) \in A \oplus M_A$,

$$\begin{tikzcd} A \oplus M_A \arrow[r, "\gamma^1_{AB}"] \arrow[d] & B \oplus (B \otimes_A M_A) \arrow[d] \\ A' \oplus M_{A'} \arrow[r, "\gamma^1_{A'B'}"] & B' \oplus (B' \otimes_{A'} M_{A'}) \end{tikzcd} \qquad \begin{tikzcd} (a, m) \arrow[r, mapsto] \arrow[dd, mapsto] & (\pi^{\#} a, 1 \otimes m) \arrow[d, mapsto] \\ & (\pi^{\#} a|_{B'},\, (1 \otimes m)|_{B'}) \arrow[d, equal] \\ (a|_{A'},\, m|_{A'}) \arrow[r, mapsto] & (\pi^{\#}(a|_{A'}),\, 1 \otimes (m|_{A'})) \end{tikzcd}$$

r ≥ 2: By induction, suppose that for each affine open $\operatorname{Spec} A \subset X$ and each affine open $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$, morphisms of abelian groups $\gamma^{r-1}_{AB} : \mathcal{P}^{r-1}_{\phi_X}(\operatorname{Spec} A) \to \mathcal{P}^{r-1}_{\phi_Y}(\operatorname{Spec} B)$ compatible with restrictions have been constructed.

Let $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset \pi^{-1} \operatorname{Spec} A$ be open affine sets. We have

$$\mathcal{P}^r_{\phi_X}(\operatorname{Spec} A) = A \oplus \left(R^{r-1}_A \,{}_{\mathrm{II}}\!\otimes M_A \right) \qquad \text{and}$$
$$\mathcal{P}^r_{\phi_Y}(\operatorname{Spec} B) \cong B \oplus \left(S^{r-1}_B \,{}_{\mathrm{II}}\!\otimes_A M_A \right).$$

Define

$$\gamma^r_{AB} : \mathcal{P}^r_{\phi_X}(\operatorname{Spec} A) \to \mathcal{P}^r_{\phi_Y}(\operatorname{Spec} B) \qquad \text{by} \qquad \gamma^r_{AB} : (a,\, t \otimes m) \mapsto (\pi^{\#} a,\, \gamma^{r-1}_{AB}(t) \otimes m).$$

The maps γ^r_{AB} are compatible with restrictions: let $\operatorname{Spec} A' \subset \operatorname{Spec} A$ and $\operatorname{Spec} B' \subset \operatorname{Spec} B$

with $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ and $\operatorname{Spec} B' \subset \pi^{-1}(\operatorname{Spec} A')$. Then for $(a, t \otimes m) \in A \oplus (R_A^{r-1} {}_{\mathrm{II}}\otimes M_A)$,

$$\begin{tikzcd} A \oplus (R_A^{r-1} {}_{\mathrm{II}}\otimes M_A) \arrow[r, "\gamma^r_{AB}"] \arrow[d] & B \oplus (S_B^{r-1} {}_{\mathrm{II}}\otimes_A M_A) \arrow[d] \\ A' \oplus (R_{A'}^{r-1} {}_{\mathrm{II}}\otimes M_{A'}) \arrow[r, "\gamma^r_{A'B'}"] & B' \oplus (S_{B'}^{r-1} {}_{\mathrm{II}}\otimes_{A'} M_{A'}) \end{tikzcd} \qquad \begin{tikzcd} (a, t \otimes m) \arrow[r, mapsto] \arrow[dd, mapsto] & (\pi^{\#} a, \gamma^{r-1}_{AB}(t) \otimes m) \arrow[d, mapsto] \\ & (\pi^{\#} a|_{B'}, (\gamma^{r-1}_{AB}(t) \otimes m)|_{B'}) \arrow[d, equal] \\ (a|_{A'}, t|_{A'} \otimes m|_{A'}) \arrow[r, mapsto] & (\pi^{\#}(a|_{A'}), \gamma^{r-1}_{A'B'}(t|_{A'}) \otimes (m|_{A'})) \end{tikzcd}$$

as γ^{r-1}_{AB} are compatible with restrictions by the induction hypothesis.

This completes the inductive construction.

By Lemma 2.16, for each $r \geq 0$, we obtain a morphism of sheaves of abelian groups $\gamma^r : \pi^{-1}\mathcal{P}^r_{\phi_X} \to \mathcal{P}^r_{\phi_Y}$ (satisfying properties (ii)(a) and (ii)(b) of the lemma).

Proposition 2.18. *The collection of morphisms γ^r constructed above is compatible with projection maps, in the sense that for any $r \geq 1$, the following square is commutative:*

$$\begin{tikzcd} \pi^{-1}\mathcal{P}^r_{\phi_X} \arrow[r, "\gamma^r"] \arrow[d, "\pi^r_{\phi_X}"'] & \mathcal{P}^r_{\phi_Y} \arrow[d, "\pi^r_{\phi_Y}"] \\ \pi^{-1}\mathcal{P}^{r-1}_{\phi_X} \arrow[r, "\gamma^{r-1}"] & \mathcal{P}^{r-1}_{\phi_Y} \end{tikzcd}$$

Proof. By Lemma 2.17, it is enough to check that for each open affine $\operatorname{Spec} A \subset X$ and each open affine $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$, the diagram

$$\begin{tikzcd} \mathcal{P}^r_{\phi_X}(\operatorname{Spec} A) \arrow[r, "\gamma^r_{AB}"] \arrow[d, "\pi^r_{\phi_X}"'] & \mathcal{P}^r_{\phi_Y}(\operatorname{Spec} B) \arrow[d, "\pi^r_{\phi_Y}"] \\ \mathcal{P}^{r-1}_{\phi_X}(\operatorname{Spec} A) \arrow[r, "\gamma^{r-1}_{AB}"'] & \mathcal{P}^{r-1}_{\phi_Y}(\operatorname{Spec} B) \end{tikzcd}$$

commutes. We proceed by induction on r.

r = 1: Let $(a,m) \in R^1_A$. The square

$$\begin{tikzcd} A \oplus M_A \arrow[r, "\gamma^1_{AB}"] \arrow[d] & B \oplus (B \otimes_A M_A) \arrow[d] \\ A \arrow[r, "\gamma^0_{AB}"] & B \end{tikzcd} \qquad \begin{tikzcd} (a,m) \arrow[r, mapsto] \arrow[dd, mapsto] & (\pi^\# a, 1 \otimes m) \arrow[d, mapsto] \\ & \pi^\# a \arrow[d, equal] \\ a \arrow[r, mapsto] & \pi^\# a \end{tikzcd}$$

commutes, verifying the statement for $r = 1$.

r ≥ 2: Let $(a, t \otimes m) \in R^r_A$. The square

$$\begin{tikzcd} A \oplus \left(R^{r-1}_A {}_{\mathrm{II}}\!\otimes M_A\right) \arrow[r, "\gamma^r_{AB}"] \arrow[d] & B \oplus \left(S^{r-1}_B {}_{\mathrm{II}}\!\otimes_A M_A\right) \arrow[d] \\ A \oplus \left(R^{r-2}_A {}_{\mathrm{II}}\!\otimes M_A\right) \arrow[r, "\gamma^{r-1}_{AB}"] & B \oplus \left(S^{r-2}_B {}_{\mathrm{II}}\!\otimes_A M_A\right) \end{tikzcd} \qquad \begin{tikzcd} (a, t \otimes m) \arrow[r, mapsto] \arrow[dd, mapsto] & (\pi^\# a, \gamma^{r-1}_{AB}(t) \otimes m) \arrow[d, mapsto] \\ & (\pi^\# a, \pi^{r-1}\left(\gamma^{r-1}_{AB}(t)\right) \otimes m) \arrow[d, equal] \\ (a, \pi^{r-1}(t) \otimes m) \arrow[r, mapsto] & (\pi^\# a, \gamma^{r-2}_{AB}\left(\pi^{r-1}(t)\right) \otimes m) \end{tikzcd}$$

commutes. □

Proposition 2.19. *For each $r \geq 0$, γ^r is a morphism of sheaves of rings.*

Proof. By Lemma 2.16, it is enough to check that each γ^r_{AB} is a morphism of rings.

For $r = 0$, $\pi^\#$ is a morphism of sheaves of rings, hence so is its restriction to any open set. For $r = 1$, we verify that γ^1_{AB} is a ring morphism: we have $(a,m)\cdot(a',m') = (aa',\, am' + a'm)$ and

$$\begin{aligned} \gamma^1_{AB}((aa',\, am' + a'm)) &= (\pi^\#(aa'),\, 1 \otimes (am' + a'm)) \\ &= (\pi^\# a \cdot \pi^\# a',\, \pi^\# a(1 \otimes m') + \pi^\# a'(1 \otimes m)) \\ &= (\pi^\# a,\, 1 \otimes m) \cdot (\pi^\# a',\, 1 \otimes m') \\ &= \gamma^1_{AB}((a,m)) \cdot \gamma^1_{AB}((a',m')). \end{aligned}$$

For $r \geq 2$,

$$
\begin{aligned}
(a, t \otimes m) \cdot (a', t' \otimes m') &= (aa', \pi^r((a, t \otimes m)) \cdot (t' \otimes m') + \pi^r((a', t' \otimes m')) \cdot (t \otimes m)) \\
&= \Big(aa', \ \big((a, \pi^{r-1}(t) \otimes m)t'\big) \otimes m' \ + \ \big((a', \pi^{r-1}(t') \otimes m')t\big) \otimes m\Big),
\end{aligned}
$$

so that

$$
\begin{aligned}
&\gamma^r_{AB}((a, t \otimes m) \cdot (a', t' \otimes m')) \\
&= \Big(\pi^\#(aa'), \ \gamma^{r-1}_{AB}\big((a, \pi^{r-1}(t) \otimes m)t'\big) \otimes m' \ + \ \gamma^{r-1}_{AB}\big((a', \pi^{r-1}(t') \otimes m')t\big) \otimes m\Big) \\
&= \Big(\pi^\#(aa'), \ \big((\pi^\# a, \gamma^{r-2}_{AB}(\pi^{r-1}(t)) \otimes m)\gamma^{r-1}_{AB}(t')\big) \otimes m' \ + \ \big((\pi^\# a', \gamma^{r-2}_{AB}(\pi^{r-1}(t')) \otimes m')\gamma^{r-1}_{AB}(t)\big) \otimes m\Big),
\end{aligned}
$$

since γ^{r-1}_{AB} is a ring morphism by induction. On the other hand,

$$
\begin{aligned}
&\gamma^r_{AB}((a, t \otimes m)) \cdot \gamma^r_{AB}((a', t' \otimes m')) = (\pi^\# a, \ \gamma^{r-1}_{AB}(t) \otimes m) \cdot (\pi^\# a', \ \gamma^{r-1}_{AB}(t') \otimes m') \\
&= \Big(\pi^\# a \cdot \pi^\# a', \ \pi^r\big((\pi^\# a, \ \gamma^{r-1}_{AB}(t) \otimes m)\big) \cdot (\gamma^{r-1}_{AB}(t') \otimes m') + \pi^r\big((\pi^\# a', \ \gamma^{r-1}_{AB}(t') \otimes m')\big) \cdot (\gamma^{r-1}_{AB}(t) \otimes m)\Big) \\
&= \Big(\pi^\#(aa'), \ \big((\pi^\# a, \ \pi^{r-1}(\gamma^{r-1}_{AB}(t)) \otimes m)\gamma^{r-1}_{AB}(t')\big) \otimes m' \ + \ \big((\pi^\# a', \ \pi^{r-1}(\gamma^{r-1}_{AB}(t')) \otimes m')\gamma^{r-1}_{AB}(t)\big) \cdot \otimes m\Big).
\end{aligned}
$$

We see that if $\pi^{r-1}(\gamma^{r-1}_{AB}(t)) = \gamma^{r-2}_{AB}(\pi^{r-1}(t))$,

$$
\gamma^r_{AB}((a, t \otimes m)) \cdot \gamma^r_{AB}((a', t' \otimes m')) = \gamma^r_{AB}((a, t \otimes m)(a', t' \otimes m'))
$$

(and similar equality for t'). But the former is true because γ^r_{AB} are compatible with restrictions (Proposition 2.18). $\square$

Proposition 2.20. *Let $r \geq 0$ be an integer. The morphism γ^r is compatible with ψ_{I} and ψ_{II}, in the sense that the following two squares commute:*

$$
\begin{tikzcd}
\pi^{-1}\mathcal{P}^r_{\phi_X} \arrow[r, "\gamma^r"] & \mathcal{P}^r_{\phi_Y} \\
\pi^{-1}\mathcal{O}_X \arrow[u, "\pi^{-1}\psi^r_{\mathrm{I}}"] \arrow[r, "\pi^\#"'] & \mathcal{O}_Y \arrow[u, "\psi^r_{\mathrm{I}}"']
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\pi^{-1}\mathcal{P}^r_{\phi_X} \arrow[r, "\gamma^r"] & \mathcal{P}^r_{\phi_Y} \\
\pi^{-1}\mathcal{O}_X \arrow[u, "\pi^{-1}\psi^r_{\mathrm{II}}"] \arrow[r, "\pi^\#"'] & \mathcal{O}_Y \arrow[u, "\psi^r_{\mathrm{II}}"']
\end{tikzcd}
$$

Proof. By Lemma 2.17, it is enough to check on open affines.

Let $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$ be open affines. We recall the compatibility hypothesis (2.16): $\pi^*\phi_X = \phi_Y \circ d\pi$, so that also $\pi^* D_X = D_Y \circ \pi^*$, where $D_X := \phi_X \circ d_X$ and $D_Y = \phi_Y \circ d_Y$.

For $r = 0$, $\gamma^0 = \pi^\#$ and $\psi^0_{\mathrm{I}} = \psi^0_{\mathrm{II}} = \mathrm{id}$, so that there is nothing to check.

For $r = 1$, the rings of sections are

$$\begin{tikzcd}
A \oplus M_A \arrow[r] & B \oplus (B \otimes_A M_A) \\
A \arrow[u] \arrow[r] & B \arrow[u]
\end{tikzcd}$$

An element $a \in A$ gets sent to

$$\begin{tikzcd}
(a, 0) \arrow[r, mapsto] & (\pi^\# a,\, 0) \arrow[d, equal] \\
 & (\pi^\# a,\, 0) \\
a \arrow[uu, mapsto] \arrow[r] & \pi^\# a \arrow[u, mapsto]
\end{tikzcd} \quad \text{by } \psi^1_{\mathrm{I}} \qquad \text{and} \qquad
\begin{tikzcd}
(a, D_X(a)) \arrow[r, mapsto] & (\pi^\# a,\, 1 \otimes D_X(a)) \arrow[d, equal] \\
 & (\pi^\# a,\, 1 \otimes D_Y(\pi^\# a)) \\
a \arrow[uu, mapsto] \arrow[r] & \pi^\# a \arrow[u, mapsto]
\end{tikzcd} \quad \text{by } \psi^1_{\mathrm{II}}$$

For $r \geq 2$, the rings of sections are

$$\begin{tikzcd}
A \oplus (R^{r-1}_A \,{}_{\mathrm{II}}\!\otimes_A M_A) \arrow[r, "\gamma^r_{AB}"] & B \oplus (S^{r-1}_B \,{}_{\mathrm{II}}\!\otimes_B M_A) \\
A \arrow[u] \arrow[r] & B \arrow[u]
\end{tikzcd}$$

An element $a \in A$ gets sent to

$$\begin{tikzcd}
(a,\, 0) \arrow[r, mapsto] & (\pi^\# a,\, 0) \arrow[d, equal] \\
 & (\pi^\# a,\, 0) \\
a \arrow[uu, mapsto] \arrow[r, mapsto] & \pi^\# a \arrow[u, mapsto]
\end{tikzcd} \quad \text{by } \psi^r_{\mathrm{I}} \quad \text{and} \quad
\begin{tikzcd}
(a,\, 1 \otimes D_X(a)) \arrow[r, mapsto] & (\pi^\# a,\, \gamma^{r-1}(1) \otimes D_X(a)) \arrow[d, equal] \\
 & (\pi^\# a,\, 1 \otimes D_Y(\pi^\# a)) \\
a \arrow[uu, mapsto] \arrow[r, mapsto] & \pi^\# a \arrow[u, mapsto]
\end{tikzcd} \quad \text{by } \psi^r_{\mathrm{II}}$$

$\square$

The next goal is to show that the maps γ^r induce isomorphisms between $\pi^*\mathcal{P}^r_{\phi_X}$ and

$\mathcal{P}^r_{\phi_Y}$, where both vector bundles either both have their ψ_{I} or both have their ψ_{II} $\mathcal{O}_X$-module structure.

We proceed similarly to before, beginning with a lemma that will be useful for reducing global statements to computations over affine open sets.

Lemma 2.21. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be ringed spaces, $\pi : Y \to X$ be a morphism of ringed spaces, $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for the topologies of X and Y, respectively, and $\mathcal{F}$ and $\mathcal{G}$ be an $\mathcal{O}_X$-algebra and an $\mathcal{O}_Y$-algebra, respectively.*

For each pair of open sets $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$ with $U \subset \pi^{-1}(V)$, let γ_{UV} be a map of rings

$$\gamma_{UV} : \mathcal{F}(U) \to \mathcal{G}(V)\,,$$

and suppose that the collection $\{\gamma_{UV}\}_{U \in \mathcal{B}_X, V \in \mathcal{B}_Y, U \subset \pi^{-1}(V)}$ is compatible with restrictions. Suppose, moreover, that for each pair of open sets $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$ with $U \subset \pi^{-1}(V)$, the square

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\gamma_{UV}} & \mathcal{G}(V) \\ \uparrow & & \uparrow \\ \mathcal{O}_X(U) & \xrightarrow[\pi^{\#}]{} & \mathcal{O}_Y(V) \end{array} \tag{2.24}$$

(where the two vertical maps are the algebra structure maps) is commutative.

As usual, denote the pullback $\pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_Y$ by $\pi^*\mathcal{F}$. We remind the reader that for each $y \in Y$, there is a canonical identification

$$(\pi^*\mathcal{F})_y = (\pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_Y)_y = \mathcal{F}_{\pi(y)} \otimes_{\mathcal{O}_{X,\pi(y)}} \mathcal{O}_{Y,y}.$$

With the above set-up, there is a unique morphism of $\mathcal{O}_Y$-algebras

$$\gamma : \pi^*\mathcal{F} \to \mathcal{G}$$

that satisfies the property that for each point y of Y, and each pair of open sets $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$

with $y \in V \subset \pi^{-1}(U)$, the square

$$\begin{array}{ccc} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V) & \longrightarrow & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}_{\pi(y)} \otimes_{\mathcal{O}_{X,\pi(y)}} \mathcal{O}_{Y,y} & \xrightarrow[\gamma_y]{} & \mathcal{G}_y \end{array} \tag{2.25}$$

commutes (where the top horizontal map is obtained by the universal property of the pushout from the diagram (2.24) corresponding to U and V).

Proof. By Lemma 2.16, there is a unique morphism $\gamma' : \pi^{-1}\mathcal{F} \to \mathcal{G}$ of sheaves of rings on Y, with the property that for each $y \in Y$, and each $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$ with $y \in V \subset \pi^{-1}(U)$, the following square commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\gamma_{UV}} & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}_{\pi(y)} & \xrightarrow[\gamma'_y]{} & \mathcal{G}_y \end{array}$$

By Lemma 2.17 and commutativity of (2.24), the induced morphisms $\gamma' : \pi^{-1}\mathcal{F} \to \mathcal{G}$ and $\pi^{\#} : \pi^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ make the square

$$\begin{array}{ccc} \pi^{-1}\mathcal{F} & \xrightarrow{\gamma'} & \mathcal{G} \\ \uparrow & & \uparrow \\ \pi^{-1}\mathcal{O}_X & \xrightarrow[\pi^{\#}]{} & \mathcal{O}_Y \end{array}$$

commute (where the vertical maps are the two algebra structure morphisms). The commutativity of the above square implies that γ' is a morphism of $\mathcal{O}_Y$-algebras.

Because the tensor product $\pi^*\mathcal{F} = \pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_Y$ satisfies the universal property of the pushout of the diagram of $\mathcal{O}_Y$-algebras

$$\begin{array}{ccc} \pi^{-1}\mathcal{F} & & \\ \uparrow & & \\ \pi^{-1}\mathcal{O}_X & \xrightarrow[\pi^{\#}]{} & \mathcal{O}_Y \end{array}$$

there is a unique induced map $\gamma\colon \pi^*\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_Y$-algebras making the following diagram commute:

$$\begin{tikzcd}
 & & \mathcal{G} \\
\pi^{-1}\mathcal{F} \arrow[rru, bend left, "\gamma'"] \arrow[r] & \pi^*\mathcal{F} \arrow[ru, dashed, "\gamma", "\exists!"'] & \\
\pi^{-1}\mathcal{O}_X \arrow[u] \arrow[r, "\pi^\#"'] & \mathcal{O}_Y \arrow[u] \arrow[ruu, bend right] &
\end{tikzcd}$$

and for each $y \in Y$, there is a corresponding commuting diagram of stalks

$$\begin{tikzcd}
 & & \mathcal{G}_y \\
\mathcal{F}_{\pi(y)} \arrow[rru, bend left, "\gamma_y'"] \arrow[r] & \mathcal{F}_{\pi(y)} \otimes_{\mathcal{O}_{X,\pi(y)}} \mathcal{O}_{Y,y} \arrow[ru, "\gamma_y"] & \\
\mathcal{O}_{X,\pi(y)} \arrow[u] \arrow[r, "\pi_y^\#"'] & \mathcal{O}_{Y,y} \arrow[u] \arrow[ruu, bend right] &
\end{tikzcd}$$

Fix $y \in Y$, and $U \in \mathcal{B}_X$, $V \in \mathcal{B}_Y$ with $y \in V \subset \pi^{-1}(U)$. Denote the structure morphism $\mathcal{O}_Y \to \mathcal{G}$ by τ. We know that the following two squares are commutative:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\gamma_{UV}"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\
\mathcal{F}_{\pi(y)} \arrow[r, "\gamma_y'"'] & \mathcal{G}_y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "\tau(V)"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\
\mathcal{O}_{Y,y} \arrow[r, "\tau_y"'] & \mathcal{G}_y
\end{tikzcd}$$

Moreover, by construction of $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V) \to \mathcal{G}(V)$, the following diagram commutes:

$$\begin{tikzcd}
 & & \mathcal{G}(V) \\
\mathcal{F}(U) \arrow[rru, bend left, "\gamma_{UV}"] \arrow[r] & \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V) \arrow[ru] & \\
\mathcal{O}_X(U) \arrow[u] \arrow[r, "\pi^\#"] & \mathcal{O}_Y(V) \arrow[u] \arrow[ruu, bend right, "\tau(V)"'] &
\end{tikzcd}$$

Let $f \otimes g$, $f \in \mathcal{F}(U)$, $g \in \mathcal{O}_Y(V)$, be a typical generator of $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V)$. Chasing

$(f,U)\otimes(g,V)$ through the square

$$\begin{tikzcd}
\mathcal{F}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{O}_Y(V) \arrow[r] \arrow[d] & \mathcal{G}(V) \arrow[d] \\
\mathcal{F}_{\pi(y)}\otimes_{\mathcal{O}_{X,\pi(y)}}\mathcal{O}_{Y,y} \arrow[r, "\gamma_y"] & \mathcal{G}_y
\end{tikzcd}$$

we have

$$\begin{tikzcd}
f\otimes g \arrow[r, mapsto] \arrow[dd, mapsto] & \gamma_{UV}(f)\cdot\tau(V)(g) \arrow[d, mapsto] \\
& (\gamma_{UV}(f)\cdot\tau(V)(g),\,V) \arrow[d, equal] \\
(f,U)\otimes(g,V) \arrow[r, mapsto] & g'_y((f,U))\cdot\tau_y((g,V)) = (\gamma_{UV}(f),V)\cdot(\tau(V)(g),V)
\end{tikzcd}$$

so that the square commutes.

This completes the verification that γ exists, and that, for every $y\in Y$ and $U\in\mathcal{B}_X$, $V\in\mathcal{B}_Y$ with $y\in V\subset\pi^{-1}(U)$, the corresponding square (2.25) commutes.

We argue that γ is uniquely characterized. Suppose that $\phi\colon\pi^*\mathcal{F}\to\mathcal{G}$ is a morphism of $\mathcal{O}_Y$-algebras such that that for every $y\in Y$ and $U\in\mathcal{B}_X$, $V\in\mathcal{B}_Y$ with $y\in V\subset\pi^{-1}(U)$ the square

$$\begin{tikzcd}
\mathcal{F}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{O}_Y(V) \arrow[r] \arrow[d] & \mathcal{G}(V) \arrow[d] \\
\mathcal{F}_{\pi(y)}\otimes_{\mathcal{O}_{X,\pi(y)}}\mathcal{O}_{Y,y} \arrow[r, "\phi_y"'] & \mathcal{G}_y
\end{tikzcd}$$

commutes.

Then, for every $y\in Y$, we have $\phi_y=\gamma_y$ (for any germ g in $\mathcal{F}_{\pi(y)}\otimes_{\mathcal{O}_{X,\pi(y)}}\mathcal{O}_{Y,y}$, there is a choice of base open sets $U\subset X$, $y\in V\subset\pi^{-1}(U)$ so that g is in the image of the map $\mathcal{F}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{O}_Y(V)\to\mathcal{F}_{\pi(y)}\otimes_{\mathcal{O}_{X,\pi(y)}}\mathcal{O}_{Y,y}$), so that $\phi=\gamma$. □

We introduce the notation

$$\pi_{\mathrm{I}}^*\mathcal{P}^r_{\phi_X}=\pi^{-1}\mathcal{P}^r_{\phi_X\ \mathrm{I}}\otimes_{\pi^{-1}\mathcal{O}_X}\mathcal{O}_Y \qquad\text{and}\qquad \pi_{\mathrm{II}}^*\mathcal{P}^r_{\phi_X}=\pi^{-1}\mathcal{P}^r_{\phi_X\ \mathrm{II}}\otimes_{\pi^{-1}\mathcal{O}_X}\mathcal{O}_Y.$$

By the above lemma, Proposition 2.19 and (the proof of) Proposition 2.20, we obtain morphisms of $\mathcal{O}_Y$-algebras

$$\gamma_{\mathrm{I}}^r : \pi_{\mathrm{I}}^*\mathcal{P}_{\phi_X}^r \to \mathcal{P}_{\phi_Y}^r \qquad \text{and} \qquad \gamma_{\mathrm{II}}^r : \pi_{\mathrm{II}}^*\mathcal{P}_{\phi_X}^r \to \mathcal{P}_{\phi_Y}^r$$

(where in the left equality $\mathcal{P}_{\phi_Y}^r$ is taken with its ψ_{I} $\mathcal{O}_Y$-module structure, and in the right equality $\mathcal{P}_{\phi_Y}^r$ is taken with its ψ_{II} $\mathcal{O}_Y$-module structure).

Proposition 2.22. *For each $r \geq 1$, the following two diagrams of $\mathcal{O}_Y$-modules commute:*

$$\begin{tikzcd} 0 \arrow[r] & \pi^*\left(M^{\otimes r}\right) \arrow[r] \arrow[d] & \pi_{\mathrm{I}}^*\mathcal{P}_{\phi_X}^r \arrow[r, "\pi_{\mathrm{I}}^*\pi_X^r"] \arrow[d, "\gamma_{\mathrm{I}}^r"] & \pi_{\mathrm{I}}^*\mathcal{P}_{\phi_X}^{r-1} \arrow[r] \arrow[d, "\gamma_{\mathrm{I}}^{r-1}"] & 0 \\ 0 \arrow[r] & \left(\pi^*M\right)^{\otimes r} \arrow[r] & \mathcal{P}_{\phi_Y}^r \arrow[r, "\pi_Y^r"] & \mathcal{P}_{\phi_Y}^{r-1} \arrow[r] & 0 \end{tikzcd} \tag{2.26}$$

and

$$\begin{tikzcd} 0 \arrow[r] & \pi^*\left(M^{\otimes r}\right) \arrow[r] \arrow[d] & \pi_{\mathrm{II}}^*\mathcal{P}_{\phi_X}^r \arrow[r, "\pi_{\mathrm{II}}^*\pi_X^r"] \arrow[d, "\gamma_{\mathrm{II}}^r"] & \pi_{\mathrm{II}}^*\mathcal{P}_{\phi_X}^{r-1} \arrow[r] \arrow[d, "\gamma_{\mathrm{II}}^{r-1}"] & 0 \\ 0 \arrow[r] & \left(\pi^*M\right)^{\otimes r} \arrow[r] & \mathcal{P}_{\phi_Y}^r \arrow[r, "\pi_Y^r"] & \mathcal{P}_{\phi_Y}^{r-1} \arrow[r] & 0 \end{tikzcd} \tag{2.27}$$

(where, in both diagrams, $\pi^*\left(M^{\otimes r}\right) \to \left(\pi^*M\right)^{\otimes r}$ is the natural isomorphism).

Proof. We prove that the diagram (2.27) is commutative for each $r \geq 1$; the proof that the diagram (2.26) is commutative is similar (replacing II by I everywhere it is necessary).

Let $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A) \subset Y$ be open affines. As before, use the notation

$$M_A = M(\operatorname{Spec} A), \qquad R_A^r = \mathcal{P}_{\phi_X}^r(\operatorname{Spec} A), \qquad S_B^r = \mathcal{P}_{\phi_Y}^r(\operatorname{Spec} B).$$

We remind the reader that above, for each $r \geq 0$, we have constructed a morphism of rings

$$\gamma^r_{AB} \colon R^r_A \to S^r_B.$$

Moreover, the proof of Proposition 2.18 shows that, for each $r \geq 1$, the square

$$\begin{tikzcd} R^r_A \arrow[r, "\gamma^r_{AB}"] \arrow[d, "\pi^r"'] & S^r_B \arrow[d, "\pi^r"] \\ R^{r-1}_A \arrow[r, "\gamma^{r-1}_{AB}"'] & S^{r-1}_B \end{tikzcd} \tag{2.28}$$

commutes.

The map

$$\mathcal{P}^r_{\phi_X}(\operatorname{Spec} A) \otimes_{\mathcal{O}_X(\operatorname{Spec} A)} \mathcal{O}_Y(\operatorname{Spec} B) = R^r_A \otimes_A B \to S^r_B = \mathcal{P}^r_{\phi_Y}(\operatorname{Spec} B)$$

obtained by the universal property of a pushout is given by

$$t \otimes b \mapsto \psi^r_{\mathrm{II}}(b)\gamma^r_{AB}(t).$$

For each $r \geq 1$, we have natural isomorphisms

$$\begin{aligned}(M_A \otimes_A B)^{\otimes_B r} &= (M_A \otimes_A B) \otimes_B (M_A \otimes_A B) \otimes_B \cdots \otimes_B (M_A \otimes_A B) \\ &\cong (M_A \otimes_A \cdots \otimes_A M_A) \otimes_A B \\ &= ({M_A}^{\otimes_A r}) \otimes_A B.\end{aligned}$$

Under the above isomorphisms, a typical generator $(m_1 \otimes b_1) \otimes \cdots \otimes (m_r \otimes b_r)$ is sent to $(m_1 \otimes \cdots \otimes m_r) \otimes (b_1 \cdots b_r)$.

By Proposition 2.2, we know that for each $r \geq 1$, $\ker(\pi^r_X) \cong M^{\otimes r}$ as $\mathcal{O}_X$-modules. The proof of Proposition 2.2 in fact shows that $\ker(\pi_X^r) \cong \ker(\pi_X^{r-1}) \otimes_{\mathcal{O}_X} M$. Let

$$K_A^r := \ker(\pi_X^r)(\operatorname{Spec} A) \qquad \text{and} \qquad Q_B^r := \ker(\pi_Y^r)(\operatorname{Spec} B).$$

Then $K_A^r \cong K_A^{r-1}\,{}_{\mathrm{II}}\otimes_A M_A$ and $Q_B^r \cong Q_B^{r-1}\,{}_{\mathrm{II}}\otimes_B (M_A \otimes_A B)$. Again, by the proof of Proposition 2.2, we see that the inclusion of $K_A^{r-1}\,{}_{\mathrm{II}}\otimes_A M_A$ into $R_A^r = A \oplus (R_A^{r-1}\,{}_{\mathrm{II}}\otimes_A M_A)$ is given by $t \otimes m \mapsto (0, t \otimes m)$, and similarly the inclusion of $Q_B^{r-1}\,{}_{\mathrm{II}}\otimes_B (M_A \otimes_A B)$ into S_B^r is given by $s \otimes (m \otimes b) \mapsto (0, s \otimes (m \otimes b))$.

We verify that the right square in the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & (K_A^{r-1}\,{}_{\mathrm{II}}\otimes_A M_A) \otimes_A B \arrow[r] \arrow[d, dashed] & R_A^r \otimes_A B \arrow[r, "\pi^r\otimes \mathrm{id}_B"] \arrow[d] & R_A^{r-1} \otimes_A B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_B^{r-1}\,{}_{\mathrm{II}}\otimes_B (M_A \otimes_A B) \arrow[r] & S_B^r \arrow[r] & S_B^{r-1} \arrow[r] & 0
\end{tikzcd}$$

Indeed,

$$\begin{tikzcd}
R_A^r \otimes_A B \arrow[r, "\pi^r\otimes \mathrm{id}_B"] \arrow[d] & R_A^{r-1} \otimes_A B \arrow[d] \\
S_B^r \arrow[r] & S_B^{r-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
t \otimes b \arrow[r, mapsto] \arrow[dd, mapsto] & \pi^r(t) \otimes b \arrow[d, mapsto] \\
 & \psi_{\mathrm{II}}^{r-1}(b)\gamma_{AB}^{r-1}(\pi^r(t)) \arrow[d, equal] \\
\psi_{\mathrm{II}}^r(b)\gamma_{AB}^r(t) \arrow[r, mapsto] & \pi^r(\psi_{\mathrm{II}}^r(b)\gamma_{AB}^r(t))
\end{tikzcd}$$

where

$$\psi_{\mathrm{II}}^{r-1}(b)\gamma_{AB}^{r-1}(\pi^r(t)) = \psi_{\mathrm{II}}^{r-1}(b)\pi^r(\gamma_{AB}^r(t))$$

by diagram (2.28) and

$$\psi_{\mathrm{II}}^{r-1}(b)\pi^r(\gamma_{AB}^r(t)) = \pi^r\left(\psi_{\mathrm{II}}^r(b)\gamma_{AB}^r(t)\right)$$

by the fact that π_Y^r is a morphism of $\mathcal{O}_Y$-algebras (Proposition 2.1).

Therefore, there is an induced map of B-modules

$$(K_A^{r-1}\ {}_{\mathrm{II}}{\otimes}_A M_A) \otimes_A B \to Q_B^{r-1}\ {}_{\mathrm{II}}{\otimes}_B (M_A \otimes_A B)$$

sending $(t \otimes m) \otimes b \mapsto (\psi_{\mathrm{II}}^{r-1}(b)t) \otimes (m \otimes 1) = t \otimes (m \otimes b)$. After identifying $K^{r-1}\ {}_{\mathrm{II}}{\otimes}_A M_A$ with ${M_A}^{\otimes_A r}$ and $Q_B^{r-1}\ {}_{\mathrm{II}}{\otimes}_B (M_A \otimes_A B)$ with $(M_A \otimes_A B)^{\otimes_B r}$, this map coincides with the natural isomorphism described above.

Therefore, for every affine open $\operatorname{Spec} A \subset X$ and every affine open $\operatorname{Spec} B \subset \pi^{-1}(\operatorname{Spec} A)$, we have shown that the diagram

$$\begin{tikzcd}
0 \arrow[r] & ({M_A}^{\otimes_A r}) \otimes_A B \arrow[r] \arrow[d, "\cong"'] & R_A^r \otimes_A B \arrow[r, "\pi^r \otimes \mathrm{id}_B"] \arrow[d] & R_A^{r-1} \otimes_A B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (M_A \otimes_A B)^{\otimes_B r} \arrow[r] & S_B^r \arrow[r] & S_B^{r-1} \arrow[r] & 0
\end{tikzcd}$$

commutes. By diagram (2.25), this is sufficient to imply that for every $y \in Y$, the diagram of stalks

$$\begin{tikzcd}
0 \arrow[r] & (\pi^*(M^{\otimes r}))_y \arrow[r] \arrow[d] & \left(\pi_{\mathrm{II}}^* \mathcal{P}_{\phi_X}^r\right)_y \arrow[r] \arrow[d] & \left(\pi_{\mathrm{II}}^* \mathcal{P}_{\phi_X}^{r-1}\right)_y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \left((\pi^* M)^{\otimes r}\right)_y \arrow[r] & \left(\mathcal{P}_{\phi_Y}^r\right)_y \arrow[r] & \left(\mathcal{P}_{\phi_Y}^{r-1}\right)_y \arrow[r] & 0
\end{tikzcd}$$

commutes. Therefore, diagram (2.27) commutes as well. □

Corollary 2.23. *For each $r \geq 0$, γ_{I}^r and γ_{II}^r are isomorphisms of $\mathcal{O}_Y$-algebras.*

Proof. We prove that γ_{II}^r is an isomorphism of $\mathcal{O}_Y$-algebras for each $r \geq 0$; the proof for γ_{I}^r is similar.

First, we remind the reader that short exact sequences of locally free sheaves pull back to short exact sequences of locally free sheaves. Indeed, let $\pi : Y \to X$ be a morphism of schemes, and suppose that

$$0 \to \mathcal{E} \to \mathcal{K} \to \mathcal{L} \to 0$$

is a short exact sequence of locally free sheaves on X. Since pullback is a tensor product, we have the long exact sequence of Tor

$$\cdots \to \mathrm{Tor}_1^{\pi^{-1}\mathcal{O}_X}(\mathcal{O}_Y, \pi^{-1}\mathcal{K}) \to \mathrm{Tor}_1^{\pi^{-1}\mathcal{O}_X}(\mathcal{O}_Y, \pi^{-1}\mathcal{L}) \to \pi^*\mathcal{E} \to \pi^*\mathcal{K} \to \pi^*\mathcal{L} \to 0.$$

But $\mathrm{Tor}_1^{\pi^{-1}\mathcal{O}_X}(\mathcal{O}_Y, \pi^{-1}\mathcal{L}) = 0$, since $\pi^{-1}\mathcal{L}$ is a locally free $\pi^{-1}\mathcal{O}_X$-module, hence flat.

Therefore, by Corollary 2.3 the (commutative) diagram (2.27) has exact rows.

We now proceed by induction on r. For $r = 0$, the statement is clear, since $\pi_{\mathrm{II}}^*\mathcal{P}^0_{\phi_X} = \mathcal{O}_Y$ and $\gamma^0_{\mathrm{II}} = \mathrm{id}_{\mathcal{O}_Y}$. For $r = 1$, the diagram (2.27) specializes to

$$\begin{tikzcd}
0 \arrow[r] & \pi^*M \arrow[r] \arrow[d, "\cong"] & \pi_{\mathrm{II}}^*\mathcal{P}^1_{\phi_X} \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi^*M \arrow[r] & \mathcal{P}^1_{\phi_Y} \arrow[r] & \mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

The left and right vertical maps are isomorphisms, hence the middle vertical map is also an isomorphism by the five lemma.

Now let $r \geq 2$. In this case the diagram (2.27) specializes to

$$\begin{tikzcd}
0 \arrow[r] & \pi^*(M^{\otimes r}) \arrow[r] \arrow[d, "\cong"] & \pi_{\mathrm{II}}^*\mathcal{P}^r_{\phi_X} \arrow[r] \arrow[d] & \pi_{\mathrm{II}}^*\mathcal{P}^{r-1}_{\phi_X} \arrow[r] \arrow[d, "\cong \text{ by induction}"] & 0 \\
0 \arrow[r] & (\pi^*M)^{\otimes r} \arrow[r] & \mathcal{P}^r_{\phi_Y} \arrow[r] & \mathcal{P}^{r-1}_{\phi_Y} \arrow[r] & 0
\end{tikzcd}$$

The left vertical map is an isomorphism, and the right vertical map is also an isomorphism by induction. Therefore, by the five lemma the middle vertical map is an isomorphism as well. $\square$

Finally, it remains to extend the isomorphism γ^r_{I} to $\mathcal{P}^r_\phi(\mathcal{F}) = \mathcal{P}^r_\phi\,{}_{\mathrm{II}}\!\otimes \mathcal{F}$.

First, by [GD71, §0.4.3.3], we have an isomorphism

$$(\pi^{-1}\mathcal{P}^r_{\phi_X})\ {}_{\mathrm{II}}\!\otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{F} \xrightarrow{\cong} \pi^{-1}(\mathcal{P}^r_{\phi_X}\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_X} \mathcal{F})\ .$$

Moreover, the diagram

$$\begin{tikzcd}
\pi^{-1}\mathcal{O}_X \arrow[r, "\pi^{-1}(\psi^r_{\mathrm{I}} \otimes \mathrm{id})"] \arrow[d, equal] & \pi^{-1}(\mathcal{P}^r_{\phi_X}\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_X} \mathcal{F}) \\
\pi^{-1}\mathcal{O}_X \arrow[r, "\pi^{-1}(\psi^r_{\mathrm{I}}) \otimes \mathrm{id}"] & \pi^{-1}\mathcal{P}^r_{\phi_X}\ {}_{\mathrm{II}}\!\otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{F} \arrow[u, "\cong"']
\end{tikzcd}$$

commutes, which is evident upon passing to affines.

It follows that

$$\begin{aligned}
\pi_{\mathrm{I}}^*(\mathcal{P}^r_{\phi_X}\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_X} \mathcal{F}) &\cong \pi_{\mathrm{I}}^*(\mathcal{P}^r_{\phi_X})\ {}_{\mathrm{II}}\!\otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{F} \\
&\cong \pi_{\mathrm{I}}^*(\mathcal{P}^r_{\phi_X})\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{F}) \\
&= \pi_{\mathrm{I}}^*(\mathcal{P}^r_{\phi_X})\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_Y} \pi^*\mathcal{F}.
\end{aligned}$$

Composing the above isomorphisms with

$$\gamma^r_{\mathrm{I}} \otimes \mathrm{id} \colon \pi_{\mathrm{I}}^*(\mathcal{P}^r_{\phi_X})\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_Y} \pi^*\mathcal{F} \to \mathcal{P}^r_{\phi_Y}\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_Y} \pi^*\mathcal{F},$$

we obtain an isomorphism

$$\pi_{\mathrm{I}}^*(\mathcal{P}^r_{\phi_X}\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_X} \mathcal{F}) \xrightarrow{\cong} \mathcal{P}^r_{\phi_Y}\ {}_{\mathrm{II}}\!\otimes_{\mathcal{O}_Y} \pi^*\mathcal{F},$$

which we denote by $\gamma^r_{\mathrm{I}}(\mathcal{F})$.

If $V \subset H^0(X, \mathcal{F})$, we denote by π^*V the image of V under the composition

$$H^0(X,\mathcal{F}) \to H^0(\pi^{-1}(X), \pi^{-1}\mathcal{F}) \to H^0(Y, \pi^{-1}\mathcal{F}), \tag{2.29}$$

where the first morphism is a special case of (2.17), and the second morphism is extension by zero.

We then have the following theorem.

Theorem 2.24 (Functoriality)**.** *Let $\pi\colon Y \to X$ be a morphism of schemes, M be a line bundle on X, $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $\phi_X\colon \Omega_X \to M$ and $\phi_Y\colon \Omega_Y \to \pi^*M$ be derivations such that $\pi^*\phi_X = \phi_Y \circ d\pi$. Then for each integer $r \geq 0$, the map*

$$\gamma_{\mathrm{I}}^r(\mathcal{F})\colon \pi_{\mathrm{I}}^*\left(\mathcal{P}_{\phi_X}^r(\mathcal{F})\right) \to \mathcal{P}_{\phi_Y}^r(\pi^*\mathcal{F})$$

is an isomorphism of $\mathcal{O}_Y$-algebras ($\mathcal{P}_{\phi_Y}^r(\pi^*\mathcal{F})$ is taken with its ψ_{I}-induced $\mathcal{O}_Y$-algebra structure).

Moreover, let $V \subset H^0(X,\mathcal{F})$, and π^*V be defined as above. Then the diagram

$$\begin{tikzcd} \pi^*\left(V \otimes_k \mathcal{O}_X\right) \arrow[r] \arrow[d] & \pi_{\mathrm{I}}^*\left(\mathcal{P}_{\phi_X}^r(\mathcal{F})\right) \arrow[d, "\gamma_{\mathrm{I}}^r(\mathcal{F})"', "\cong"] \\ \pi^*V \otimes_k \mathcal{O}_Y \arrow[r] & \mathcal{P}_{\phi_Y}^r(\pi^*\mathcal{F}) \end{tikzcd} \tag{2.30}$$

commutes.

Proof. It remains to check that (2.30) commutes. To show this, it is sufficient to check that the corresponding diagram of stalks commutes over each point of Y. But the latter is clear. □

Similar statements hold for $r = \infty$.

2.5 $\mathcal{P}_\phi^r(E)$ in families

Define a family $\mathfrak{X}$ over a base B to be a surjective map $\pi\colon \mathfrak{X} \to B$ of schemes (in practice, the map π will often also be flat, but flatness is not necessary for the result of this section).

Let $\pi\colon \mathfrak{X} \to B$ be a family over B. We have the following exact sequence of sheaves on $\mathfrak{X}$:

$$\pi^*\Omega_B \to \Omega_{\mathfrak{X}} \to \Omega_{\mathfrak{X}/B} \to 0.$$

Let M be a line bundle on $\mathfrak{X}$, $\phi_\pi\colon \Omega_{\mathfrak{X}/B} \to M$ be a morphism of $\mathcal{O}_{\mathfrak{X}}$-modules, and define

$\phi\colon \Omega_{\mathfrak{X}} \to M$ as the following composition.

$$\begin{tikzcd} \Omega_{\mathfrak{X}/B} \arrow[r, "\phi_\pi"] & M \\ \Omega_{\mathfrak{X}} \arrow[u] \arrow[ur, dashed, "\text{definition of } \phi"'] & \end{tikzcd}$$

Let E be a locally free sheaf on $\mathfrak{X}$.

For any point $b \in B$, denote the fibre of $\mathfrak{X}$ over b by $X_b := \pi^{-1}(b)$, and let $i_b\colon X_b \to \mathfrak{X}$ be the closed embedding.

Introduce the following notation: $M|_{X_b} := M_b$, $\phi|_{X_b} := \phi_b$, $\phi_\pi|_{X_b} := (\phi_\pi)_b$, and $E|_{X_b} := E_b$.

By functoriality of pullback, the triangle

$$\begin{tikzcd} i_b^*\Omega_{\mathfrak{X}/B} \arrow[r, "i_b^*\phi_\pi"] & i_b^*M \\ i_b^*\Omega_{\mathfrak{X}} \arrow[u] \arrow[ur, "i_b^*\phi"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \Omega_{X_b} \arrow[r, "(\phi_\pi)_b"] & M_b \\ \Omega_{\mathfrak{X}}|_{X_b} \arrow[u] \arrow[ur, "\phi_b"'] & \end{tikzcd}$$

commutes, and consequently Theorem 2.24 implies the following theorem.

Theorem 2.25. *With the above set-up, for any $b \in B$, we have*

$$\mathcal{P}^r_\phi(E)\big|_{X_b} \cong \mathcal{P}^r_{(\phi_\pi)_b}(E_b).$$

2.6 Chern classes of $\mathcal{P}^r_\phi(L)$ (for L a line bundle)

We recall that, for a locally free sheaf $\mathcal{F}$,

$$c(\mathcal{F}) = c_0(\mathcal{F}) + c_1(\mathcal{F}) + \cdots$$

denotes its total Chern class.

Let X be a scheme, M and L be line bundles over X, and $\phi\colon \Omega_X \to M$ be an $\mathcal{O}_X$-linear map.

From the exact sequence

$$0 \to M^{\otimes r} \otimes L \to \mathcal{P}^r_\phi(L) \to \mathcal{P}^{r-1}_\phi(L) \to 0,$$

it follows by the Whitney formula that

$$c(\mathcal{P}^r_\phi(L)) = c(M^{\otimes r} \otimes L)\, c(\mathcal{P}^{r-1}_\phi(L)) = (1 + r\, c_1(M) + c_1(L))\, c(\mathcal{P}^{r-1}_\phi(L)).$$

Iterating, one obtains the following proposition:

Proposition 2.26. *The total Chern class of $\mathcal{P}^r_\phi(L)$ is given by*

$$c(\mathcal{P}^r_\phi(L)) = \prod_{k=0}^{r} (1 + k\, c_1(M) + c_1(L)). \tag{2.31}$$

Expanding the product, one obtains expressions for the Chern classes $c_m(\mathcal{P}^r_\phi(L))$ in terms of the first Chern classes of the line bundles M and L. The resulting formulas involve (signless) Stirling numbers of the first kind.

Stirling numbers of the first kind. In this paragraph, for the sake of completeness and for the convenience of the reader who is not familiar with Stirling numbers, we state and prove several basic facts regarding Stirling numbers of the first kind that will be used in the sequel. The reader who is familiar with this elementary material may wish to skip ahead to the following paragraph, and most readers will likely prefer to skip the proofs.

The material in this paragraph is based on [Sta12].

Let $0 \le k \le n$ be integers. The (signless) Stirling number of the first kind, denoted by $\sigma(n,k)$, is defined to be the number of permutations in the symmetric group S_n on n letters whose decomposition into disjoint cycles contains exactly k cycles.[6]

If $n < k$ or $k = 0$, set $\sigma(n,k) = 0$, except $\sigma(0,0) = 1$.

[6] We use the notation $\sigma(n,k)$ instead of the more standard $c(n,k)$ to avoid confusion with Chern classes.

For example, for any $n \geq 1$, $\sigma(n, 1)$ is the number of permutations in S_n with exactly one cycle, which is $n!/n = (n-1)!$, and $\sigma(n, n)$ is the number of permutations with exactly n cycles, which is 1 (the identity permutation).

Proposition 2.27. *The numbers $\sigma(n, k)$ satisfy the recurrence*

$$\sigma(n+1, k) = \sigma(n, k-1) + n\,\sigma(n, k) \quad \textit{for all } 1 \leq k \leq n.$$

Proof. If a permutation fixes the $(n+1)$-st letter, then $(n+1)$ is a cycle of length one in the decomposition of the permutation into disjoint cycles. The cycle $(n+1)$ can be added to $\sigma(n, k-1)$ permutations to form a permutation on $n+1$ elements with exactly k cycles.

Otherwise, the $(n+1)$-st letter is a part of a larger cycle. There are $\sigma(n, k)$ permutations on n letters with exactly k cycles, and the $(n+1)$-st letter can be insterted in n spots in the decomposition of each of these permutations into disjoint cycles, yielding $n\,\sigma(n, k)$ permutations on $n+1$ letters with exactly k cycles.

These account for all possible permutations in S_{n+1} with exactly k cycles, since the $(n+1)$-st element is either fixed or not, yielding the recurrence. $\square$

The above recurrence gives a convenient way to compute $\sigma(n,k)$. For example, the first several Stirling numbers

$$\begin{array}{ccccccccc}
 & & & & \sigma(1,1) & & & & \\
 & & & / & & \backslash & & & \\
 & & & \sigma(2,1) & & \sigma(2,2) & & & \\
 & & / & & \backslash \quad / & & \backslash & & \\
 & & \sigma(3,1) & & \sigma(3,2) & & \sigma(3,3) & & \\
 & / & & \backslash \quad / & & \backslash \quad / & & \backslash & \\
 & \sigma(4,1) & & \sigma(4,2) & & \sigma(4,3) & & \sigma(4,4) & \\
/ & & \backslash \quad / & & \backslash \quad / & & \backslash \quad / & & \backslash \\
\sigma(5,1) & & \sigma(5,2) & & \sigma(5,3) & & \sigma(5,4) & & \sigma(5,5)
\end{array}$$

are

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & / & & \backslash & & & \\
 & & & 1 & & 1 & & & \\
 & & / & & \backslash \quad / & & \backslash & & \\
 & & 2 & & 3 & & 1 & & \\
 & / & & \backslash \quad / & & \backslash \quad / & & \backslash & \\
 & 6 & & 11 & & 6 & & 1 & \\
/ & & \backslash \quad / & & \backslash \quad / & & \backslash \quad / & & \backslash \\
24 & & 50 & & 35 & & 10 & & 1
\end{array}$$

The result of the following simple computation will be applied several times.

Lemma 2.28. *Fix $n \geq 0$. We have*

$$\sum_{k=0}^{n} \sigma(n, n-k)\, s^{n-k} t^k = \sum_{k=0}^{n} \sigma(n,k)\, s^k t^{n-k} = s(s+t)(s+2t)\cdots(s+(n-1)t).$$

Proof. The first equality is a simple change of parameter of summation. To prove the second equality, note that the polynomial $F_n(s,t) := s(s+t)(s+2t)\cdots(s+(n-1)t)$ is homogeneous of degree n (being a product of n homogeneous linear terms), so can be written in the form

$$F_n(s,t) = s(s+t)(s+2t)\cdots(s+(n-1)t) = \sum_{k=0}^{n} \tau(n,k)\, s^k t^{n-k}$$

for some coefficients $\tau(n,k)$. We verify that the $\tau(n,k)$ satisfy the recurrence for $\sigma(n,k)$ of Proposition 2.27, as well as the same initial conditions, hence that $\tau(n,k) = \sigma(n,k)$.

By convention, the empty product is equal to one, hence $\tau(0,0) = 1$. Also, $\tau(n,k) = 0$ if $n < k$ (since the exponent of t is always nonnegative) or $k = 0$ (since the monomial t is not present in $F_n(s,t)$).

Since

$$\begin{aligned}\sum_{k=0}^{n+1} \tau(n+1,k) s^k t^{n+1-k} &= F_{n+1}(s,t) = (s+n\,t)F_n(s,t) = (s+n\,t)\left(\sum_{j=0}^{n} \tau(n,j) s^j t^{n-j}\right) \\ &= \sum_{j=0}^{n} \tau(n,j) s^{j+1} t^{n-j} + \sum_{j=0}^{n} n\,\tau(n,j) s^j t^{n+1-j} \\ &= \sum_{k=1}^{n+1} \tau(n,k-1) s^k t^{n+1-k} + \sum_{k=0}^{n} n\,\tau(n,k) s^k t^{n+1-k},\end{aligned}$$

it follows that

$$\tau(n+1,k) = \tau(n,k-1) + n\,\tau(n,k) \qquad \text{for all } 1 \le k \le n,$$

which completes the proof. □

Formulas for $c_m(\mathcal{P}^r_\phi(L))$. Write $\mu := c_1(M)$ and $\lambda := c_1(L)$. Applying Lemma 2.28 to (2.31), we have

$$\begin{aligned}c(\mathcal{P}^r_\phi(L)) &= (1+\lambda)(1+\lambda+\mu)\cdots(1+\lambda+r\,\mu) \\ &= \sum_{k=0}^{r+1} \sigma(r+1, r+1-k)\,(1+\lambda)^{r+1-k} \mu^k \\ &= \sum_{k=0}^{r+1} \sigma(r+1, r+1-k)\left(\sum_{j=0}^{r+1-k} \binom{r+1-k}{j} \lambda^j\right) \mu^k \\ &= \sum_{k=0}^{r+1} \sum_{j=0}^{r+1-k} \sigma(r+1, r+1-k) \binom{r+1-k}{j} \lambda^j \mu^k.\end{aligned}$$

Therefore,

Theorem 2.29. *For any $m = 0, 1, \ldots, r+1$, we have*

$$c_m(\mathcal{P}^r_\phi(L)) = \sum_{k=0}^{m} \sigma(r+1, r+1-k) \binom{r+1-k}{m-k} c_1(L)^{m-k} \, c_1(M)^k. \tag{2.32}$$

Proof. From the formula for $c(\mathcal{P}^r_\phi(L))$, we have

$$c_m(\mathcal{P}^r_\phi(L)) = \sum_{k=0}^{r+1} \sigma(r+1, r+1-k) \binom{r+1-k}{m-k} \lambda^{m-k} \mu^k,$$

and the claimed expression (with the summation terminating at $k = m$ instead of $k = r+1$) follows by noting that $\binom{r+1-k}{m-k} = 0$ for $k > m$. □

Another interpretation of the coefficients. Let S be a finite set. For $k \in \mathbb{N}$, define

$$J(S, k) \coloneqq \sum_{\substack{I \subseteq S \\ |I| = k}} \prod_{a \in I} a,$$

with the two conventions that a sum indexed by an empty set is equal to 0 (so that $J(S,k) = 0$ for $k > |S|$), and that a product indexed by an empty set is equal to 1 (so that $J(S, 0) = 1$).

When $S = \{1, \ldots, n\}$, write

$$J(n,k) \coloneqq J\left(\{1, \ldots, n\}, k\right), \qquad \text{so that} \qquad J(n,k) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = k}} \prod_{j \in I} j.$$

As special cases, we obtain

$$J(n,n) = n! \qquad \text{and} \qquad J(n,1) = 1 + 2 + \cdots + n,$$

and one can think of $J(n,k)$ with $1 \leq k \leq n$ as interpolating between these two functions.

By expanding the product $s(s+t)(s+2t)\cdots(s+(n-1)t)$, we see that

$$s(s+t)(s+2t)\cdots(s+(n-1)t) = \sum_{k=0}^{n} J(n-1, n-k) s^k t^{n-k},$$

so that, as a consequence of the second equality in the statement of Lemma 2.28,

$$J(n-1, n-k) = \sigma(n,k).$$

The above equality may also be proven by checking that $J(n-1,n-k)$ satisfy the same recurrence and initial conditions as $\sigma(n,k)$.

In terms of the functions $J(n,k)$, Proposition 2.29 becomes

$$c_m(\mathcal{P}^r_\phi(L)) = \sum_{k=0}^{m} J(r,k) \binom{r+1-k}{m-k} c_1(L)^{m-k}\, c_1(M)^k \qquad \text{for any } m = 0, 1, \ldots, r+1. \tag{2.33}$$

Chapter 3

Applications, part I: Weierstrass weight in degenerating families of curves

For this chapter, the base field is the complex numbers $\mathbb{C}$. We will use the languages of finite-type schemes over $\mathbb{C}$ and complex analytic spaces interchangeably.

3.1 A family of plane cubics degenerating to a cuspidal cubic

We begin with a simple example that is representative of one type of geometric situation that can be analyzed using this chapter's methods.

Let $\{C_t\}_{t\in\mathbb{C}} \subset \mathbb{P}^2_{[X:Y:Z]} \times \mathbb{C}_t$ be a family of plane cubic curves, with C_t defined by the equation

$$C_t = V(Y^2Z - X^3 - tZ^3).$$

The fibre over $t = 0$ is a cuspidal cubic, and a quick computation shows that the remaining fibres of the family are smooth.

Recall that a point p of a plane curve C is called a flex point if the multiplicity of the intersection of T_pC with C is ≥ 3. It is well-known that the flex points of the smooth cubic curve $C \subset \mathbb{P}^2$ with equation $F = 0$ are exactly the points of intersection of C with its so-called

Hessian curve:

$$H_C := V\left(\det\begin{pmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}\right),$$

(where F_{XY} denotes the second partial derivative of F with respect to X and Y, and similarly for the other entries of the matrix). In particular, since both C and H_C have degree 3, C has nine flex points by Bézout's theorem.

Returning to the family $\{C_t\}$, it is clear that the corresponding Hessians H_{C_t} $(t \neq 0)$ themselves form a family $\{H_{C_t}\}_{t\in\mathbb{C}^*}$. Set

$$\mathcal{C} = \{(x,t) \in \mathbb{P}^2 \times \mathbb{C} : x \in C_t\} \qquad \text{and} \qquad \mathcal{H} = \{(x,t) \in \mathbb{P}^2 \times \mathbb{C}^* : x \in H_{C_t}\},$$

and let $\overline{\mathcal{H}}$ denote the Zariski closure of $\mathcal{H}$ in $\mathbb{P}^2 \times \mathbb{C}$. Then the scheme-theoretic intersection $\mathcal{F}l := \mathcal{C} \cap \overline{\mathcal{H}}$ is a divisor in both $\mathcal{C}$ and $\overline{\mathcal{H}}$. For $t \neq 0$, $(\mathcal{F}l)_t$ consists of the nine flex points of C_t, and the fibre $(\mathcal{F}l)_0$ over $t = 0$ can be thought of as the limiting scheme of the flex points as $t \to 0$.

For a smooth fibre C_t, intersecting with H_{C_t}, we compute that the flex points are

$$\begin{array}{lll} [0:1:0], & [0:\sqrt{t}:1], & [0:-\sqrt{t}:1], \\ [\alpha:i\beta:1], & [\alpha\mu:i\beta:1], & [\alpha\mu^2:i\beta:1], \\ [\alpha:-i\beta:1], & [\alpha\mu:-i\beta:1], & [\alpha\mu^2:-i\beta:1], \end{array} \qquad \alpha = -\sqrt[3]{4t}, \quad \beta = \sqrt{3t}, \quad \mu = \exp(2\pi i/3),$$

and observe that, as $t \to 0$, each flex point except $[0:1:0]$ approaches the cusp of C_0 (with coordinates $[0:0:1]$). The scheme $\mathcal{F}l$ is then supported at two points, $[0:0:1]$ with multiplicity 8 and $[0:1:0]$ with multiplicity 1.

Choosing other single-parameter families of smooth planar cubic curves that acquire a cusp over a marked point, we find in each case[1] that exactly eight of the nine flex points tend to the cusp as the parameter tends to the marked point. In fact, this is a general

[1] If we are lucky and can carry out the computation!

phenomenon, and one beautiful explanation for why it occurs is given by the theory of Néron models. Here we can only give the following sketch.

Let C be a smooth genus one curve (without a chosen embedding for the moment). It is well-known that a choice of marked point p_0 of C induces an elliptic curve structure on C by means of the bijection $C \to \mathrm{Pic}(C)$ that sends p to $\mathcal{O}_C(p - p_0)$, and that $|3p_0|$ is a very ample linear series that gives an embedding of C into $\mathbb{P}^2$. Now, if $C \subset \mathbb{P}^2$ is a smooth cubic with a fixed embedding into $\mathbb{P}^2$, and p_f is one of the flex points of C, we have $|3p_f| = \mathcal{O}_{\mathbb{P}^2}(1)|_C$ (since the tangent line $T_{p_f}C$ meets C with multiplicity 3), and therefore the embedding by $|3p_f|$ recovers the fixed embedding. If p_f' is another flex point, $3p_f'$ is linearly equivalent to $3p_f$; then, $3\mathcal{O}_C(p_f' - p_f) = \mathcal{O}_C(3p_f' - 3p_f) = \mathcal{O}_C(0)$, so p_f' is a 3-torsion point. Conversely, a 3-torsion point is a flex point.

Normalizing C_0 and discarding the preimage of the singular locus, it turns out that we obtain the Néron model of the fibre of the family $\{C_t\}$ over the generic point (!). Discarding the singular point from the normalization of C_0, we obtain the set $\mathbb{P}^1 \setminus \{p\} = \mathbb{C}$ (where p is the preimage of the cusp point). One can show (for instance, by applying Tate's algorithm [Tat75]) that the group structure on $\mathbb{P}^1 \setminus \{p\}$ is that of the additive group $\mathbb{G}_a(\mathbb{C})$, which has only one 3-torsion point.

In the limit as $t \to 0$, a flex point of C_t goes to either a 3-torsion point, or the cusp point (which was discarded from the Néron model). Only one flex point of C_t goes into the 3-torsion point of $\mathbb{G}_a(\mathbb{C})$, and therefore the remaining eight flex points of C_t are absorbed into the cusp point.

A family of smooth planar cubic curves that acquire a node over a marked point may be analyzed similarly — after passing to the normalization, the underlying set of the group of smooth points of a node is $\mathbb{P}^1$ with two points removed, i.e. $\mathbb{C}^*$, and the group structure turns out to be that of the multiplicative group $\mathbb{G}_m(\mathbb{C})$. The multiplicative group has exactly three 3-torsion points, each of which is the limit of a single flex point, and the remaining six flex points of C_t are then absorbed into the node.

In this chapter, we will develop a method that provides another explanation for the numbers eight and six above, and applies to more general curve degenerations. Namely, we will consider a family $\{C_t\}_{t\in T} \subset S \times T$ of curves over a base scheme T, with each fibre contained in a smooth projective surface S, and with smooth general fibre. The appropriate generalization (and refinement) of the divisor $\mathcal{F}l \subset \mathcal{C}$ of flex points is the divisor $\mathcal{W}$ of Weierstrass points of a linear system on $\{C_t\}_{t\in T}$. Supposing that $\{C_t\}_{t\in T}$ acquires a singularity over a marked point $t_0 \in T$, we will describe a way (using ϕ-principal-parts-bundles) of measuring how many Weierstrass points (taken with multiplicity equal to their Weierstrass weight) are absorbed into the singular point as $t \to t_0$.

We now summarize the contents of this chapter.

The first major goal of the chapter is the extension of the notion of Weierstrass weight (of a linear series) to singular points of a curve C contained in a smooth surface S, and the demonstration of the fact that the sum of the Weierstrass weights of the points that get absorbed into a singular point in a curve degeneration is equal to the Weierstrass weight of the singular point.

In §3.2, we collect some preliminary material from duality theory.

In §3.3, we construct the morphism $\phi\colon \Omega_C \to \omega_C$ (that will be used for $\mathcal{P}^m_\phi(L)$ in this chapter). The construction of ϕ is carried out by choosing a local equation $f = 0$ of C over an open set $U \subset S$, restricting the local morphisms $\Omega_S(U) \to K_S(C)(U)$ given by

$$\omega \mapsto \omega \wedge \frac{df}{f}$$

to C, and checking that the restrictions do not depend on the choice of f, which allows gluing the restrictions to a global morphism ϕ.

In §3.4, we review the concept of a Weierstrass point of a linear series $V \subset H^0(C, L)$ of degree d on a smooth algebraic curve C, and the concept of the Weierstrass weight of a point of C, to set the stage for the generalization of these concepts to the case of a possibly singular curve in the following section. In particular, we recall that the Weierstrass weight at a point p may be computed as the order of vanishing of the determinant of the morphism

$$V \otimes_k \mathcal{O}_C \to \mathcal{P}^d(L)$$

at p (where $\mathcal{P}^d(L)$ is the usual principal parts bundle).

In §3.5, we extend the notion of Weierstrass weight from the smooth case to the singular case. By analogy with the smooth case, we define the ϕ-Weierstrass weight at any point of a (possibly singular) curve $C \subset S$ to be the order of vanishing of the determinant of the morphism

$$V \otimes_k \mathcal{O}_C \to \mathcal{P}^d_\phi(L),$$

at the point p, where $\phi\colon \Omega_C \to \omega_C$ is the morphism constructed in §3.3. We denote the ϕ-Weierstrass weight at p by $w_\phi(p)$.

Then, we show that the construction of the morphism ϕ behaves well in families. As a consequence, we show that for a family of curves $\{C_t\}_{t \in \mathbb{P}^1}$, with C_{t_0} singular for some $t_0 \in \mathbb{P}^1$ and $C_t \subset S$ for all t, and with the morphism of §3.3 constructed over C_t denoted by ϕ_t, there is a (nonempty) neighbourhood U of t_0, such that for all $t \in U$, there is a divisor $\mathcal{W} \subset S \times \mathbb{P}^1$, such that

$$\mathcal{W} \cap C_t = \sum_{p \in C_t} w_{\phi_t}(p)\, p.$$

This allows us to conclude that the amount of Weierstrass weight that gets absorbed into a singular point is equal to the ϕ-Weierstrass weight at that point, achieving the first major goal of the chapter.

The second major goal of the chapter is the demonstration of formula (3.10), which is a result of independent interest that in addition gives a convenient way of computing the Weierstrass weight at a singular point.

Formula (3.10) is proven in §3.6. Before the proof is obtained, it is necessary to develop some additional objects (some of which are of independent interest).

We therefore begin §3.6 by constructing the adjoint divisor Δ of the normalization $\pi\colon \tilde{C} \to C$ of C. By definition,

$$\Delta = \sum_{p \in C} \Delta_p,$$

where

$$\Delta_p = \sum_{r \in \pi^{-1}(p)} \gamma_r\, r,$$

and where, in turn, for any point $r \in \pi^{-1}(p)$,

$$\gamma_r = -\operatorname{ord}_r \pi^* \left(\frac{dx}{\partial f / \partial y} \right),$$

$f = 0$ being a local equation of C in a neighbourhood of p.

One reason for introducing the divisor Δ is that it allows us to identify the vector bundle obtained by pulling back the bundle $\mathcal{P}^m_\phi(L)$ to the normalization of C. Namely, we show that we have the isomorphism

$$\pi^* \mathcal{P}^m_\phi(L) \cong \mathcal{P}^m_{\cdot\Delta}(\pi^* L),$$

where $\cdot\Delta\colon K_{\tilde{C}} \to K_{\tilde{C}}(\Delta)$ is the morphism obtained by multiplying the sections of $K_{\tilde{C}}$ by a local equation of Δ.

The identification of the pullback of $\mathcal{P}^m_\phi(L)$ to the normalization of C, then, allows us to demonstrate the formula (3.10) at the end of §3.6. The statement of the formula is

$$w_\phi(p) = (\deg \Delta_p) \binom{n+1}{2} + \sum_{r \in \pi^{-1}(p)} \tilde{w}(r),$$

where $\tilde{w}(r)$ is the Weierstrass weight of the pulled back linear series $\pi^* V$ at $r \in \tilde{C}$, and the proof of this formula achieves the second major goal of the chapter.

We finish the chapter by computing several examples in §3.7. In particular, we give another proof that exactly eight of the nine flex points get absorbed into the cusp in a degeneration of a family of plane elliptic curves to a cuspidal cubic, and exactly six of the flex points get absorbed into the node in a degeneration of a family of smooth planar elliptic curves to a nodal cubic.

3.2 The dualizing sheaf and adjunction

This section contains several preliminary facts about the dualizing sheaf (all of the references are to [Har77]).

Definition. Let X be a proper scheme of dimension n over a field k. A dualizing sheaf for X is a coherent sheaf ω_X on X, together with a trace morphism $t\colon H^n(X, \omega_X) \to k$, with the property that for each coherent sheaf $\mathcal{F}$ on X,

$$\operatorname{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$$

is a perfect pairing.

Theorem. *Let X be a projective scheme over a field k.*

(i) (Propositions III.7.2 and III.7.5) The sheaf ω_X exists and is unique up to isomorphism.

(ii) (Corollary III.7.12) If X is also smooth, then $\omega_X \cong K_X = \bigwedge^{\dim X} \Omega_X$.

(iii) (Proposition III.7.5) If X is embedded as a closed subvariety of $\mathbb{P}^N_k$, then

$$\omega_X \cong \mathcal{E}xt^{\operatorname{codim}(X, \mathbb{P}^N_k)}_{\mathbb{P}^N_k}\left(\mathcal{O}_X, \Omega_{\mathbb{P}^N_k}\right) \qquad \textit{($\mathcal{E}xt$ denotes the sheaf Ext).}$$

Duality. Although the duality theorem will not be directly applied in this chapter, it was the impetus for the introduction of the dualizing sheaf, so we include its statement.

Theorem (Duality for Projective Schemes — Theorem III.7.6)**.** *Let X be an equidimensional Cohen-Macaulay projective scheme of dimension n over an algebraically closed field k. For all $i \geq 0$ and all coherent sheaves $\mathcal{F}$ on X,* $\operatorname{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$ *is a perfect pairing.*

Adjunction. For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, denote the dual sheaf of $\mathcal{F}$ by $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Let X be a closed local complete intersection of codimension r in a smooth variety Y over a field k (with the embedding denoted by $i\colon X \to Y$), $\mathcal{I}$ be the ideal sheaf of X in Y, and $\mathcal{N}_{X/Y} = (\mathcal{I}/\mathcal{I}^2)^\vee$ be the normal sheaf of the embedding.

Theorem (Adjunction, Theorem III.7.11)**.** *We have* $\omega_X \cong i^*\left(K_Y \otimes \bigwedge^r \mathcal{N}_{X/Y}\right)$.

Corollary. *For a closed embedding $i\colon C \to S$ of a curve C in a smooth projective surface S, the adjunction formula implies that*

$$\omega_C \cong i^* K_S(C).$$

In particular, ω_C is a line bundle, there is a morphism $K_S(C) \to i_*\omega_C$, and if $U \subset S$ is an open subset with local coordinates x, y in which C is given by the equation $f(x,y) = 0$, $i^*\left(\dfrac{dx \wedge dy}{f}\right)$ gives a local frame for $\omega_C(U)$.

Proof. Because C is an effective Cartier divisor in S (S being smooth), the normal sheaf of C in S is

$$\mathcal{N}_{C/S} \cong i^*\mathcal{O}_S(C).$$

Therefore, by Theorem 3.2,

$$\omega_C \cong i^*\left(K_S \otimes \mathcal{O}_S(C)\right) = i^* K_C(S).$$

Being the pullback of a line bundle, ω_C is itself a line bundle. Because i_* and i^* are adjoint functors, we have $\mathrm{Hom}_{\mathcal{O}_C}(i^*K_C(C), \omega_C) = \mathrm{Hom}_{\mathcal{O}_S}(K_C(C), i_*\omega_C)$. The statement regarding the local frame follows because $(dx \wedge dy)/f$ is a local frame for $K_S(C)$ over U.

$\square$

3.3 Construction of the derivation $\phi\colon \Omega_C \to \omega_C$

Let C be a (possibly singular) curve contained in a smooth projective surface S, and let $i\colon C \to S$ be the closed embedding.

Let U be an open neighbourhood in S with $U \cap C \neq \varnothing$, with local coordinates x and y, and local equation $f = 0$ for C. Over U, the $\mathcal{O}_S(U)$-module $\Omega_S(U)$ is generated by dx and dy, and $K_S(C)(U)$ is generated by $(dx \wedge dy)/f$. Define an $\mathcal{O}_S(U)$-module map $\Omega_S(U) \to K_S(C)(U)$ by

$$\omega \mapsto \omega \wedge \frac{df}{f}$$

or, more explicitly,

$$a\,dx + b\,dy \mapsto \left(a\,\frac{\partial f}{\partial y} - b\,\frac{\partial f}{\partial x}\right)\frac{dx \wedge dy}{f}, \qquad a,\, b \in \mathcal{O}_S(U).$$

Composing with the map $K_S(C)(U) \to i_*\omega_C(U)$ given by adjunction (Corollary in §3.2), we obtain a map $\Omega_S(U) \to i_*\omega_C(U)$, for any choice of local equation f of C in U.

Now, over U, the sections of the sheaf $i_*\Omega_C$ are given by

$$i_*\Omega_C(U) = \frac{(dx, dy)}{(df)}.$$

Since under $\Omega_S(U) \to i_*\omega_C(U)$, df gets sent to $df \wedge (df/f) = 0$, we obtain an induced map $i_*\Omega_C(U) \to i_*\omega_C(U)$.

Suppose that $g = 0$ is another local equation for C in the same open neighbourhood U with same coordinates x and y. Then $g = uf$, where u is a unit in $\mathcal{O}_S(U)$. Being a logarithmic differential, we have

$$\frac{dg}{g} = \frac{d(uf)}{uf} = \frac{u\,df + f\,du}{uf} = \frac{df}{f} + \frac{du}{u}.$$

Writing $\omega \wedge du/u$ in terms of the local frame $(dx \wedge dy)/f$ of $K_S(C)(U)$, we have

$$\left(a\frac{\partial u}{\partial y} - b\frac{\partial u}{\partial x}\right)\frac{dx \wedge dy}{u} = \left(a\frac{\partial u}{\partial y} - b\frac{\partial u}{\partial x}\right)\frac{f}{u}\frac{dx \wedge dy}{f}.$$

The last expression clearly vanishes on C, being a product of a regular section over U with f. Therefore, the map $i_*\Omega_C(U) \to i_*\omega_C(U)$ does not depend on the choice of local equation after restriction to C.

Cover C by open neighbourhoods in S with a choice of local equation in each. We have constructed a section of $\mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \omega_C)(U \cap C)$ (the sheaf Hom) for each neighbourhood U in the cover. For any intersection of two such neighbourhoods, say $U \cap U'$ with local equations f and f', respectively, the restrictions of the local equations to $U \cap U'$ are again local equations in the intersection. Since we have shown that the restriction of the map $i_*\Omega_C(U \cap U') \to i_*\omega_C(U \cap U')$ does not depend on the choice of local equation, we see that the maps $i_*\Omega_C(U) \to i_*\omega_C(U)$ and $i_*\Omega_C(U') \to i_*\omega_C(U')$ agree on the intersection. Therefore, we can glue together the sections in $\mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \omega_C)(U \cap C)$ to a section in $\mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \omega_C)(C)$, that is, a morphism $\Omega_C \to \omega_C$ of $\mathcal{O}_C$-modules.

We summarize the construction of this section by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_S \arrow[r] \arrow[d] & \Omega_S \arrow[r, dashed, "\wedge df/f", "\text{Over } U"'] \arrow[d] & K_S(C) \arrow[d, "\text{Restriction}"] \\
i_*\mathcal{O}_C \arrow[r] & i_*\Omega_C \arrow[r, "\exists"] & i_*\omega_C
\end{tikzcd}$$

Remark. When C is a smooth curve, the composition $\Omega_C \xrightarrow{\phi} \omega_C \xrightarrow{\cong} \Omega_C$ (where the first map is the derivation ϕ constructed above, and the second map is the identification of ω_C with $\Omega_C = K_C$) is the identity morphism. To see this, note that our construction of ϕ becomes the construction of the Poincaré residue map when specialized to the case when C is smooth, and the latter is an explicit isomorphism $K_S(C)|_C \cong \Omega_C$ (cf. [GH78, p. 147]).

3.4 Weierstrass points in the smooth case

The notion of a Weierstrass point of a linear series on a curve can be considered to be a refinement (along several directions) of the notion of a flex point of a plane curve — one considers arbitrary linear series (not necessarily inducing an embedding into $\mathbb{P}^2$, and even possibly with base points), one considers all of the members of the linear series instead of only the tangent lines to the image of C, and one keeps track of finer information regarding multiplicities. For plane cubic curves, the Weierstrass points are exactly the flex points. In this section, we rapidly review the basic definitions, referring to [GH78, §2.4] or [ACGH84, App. A] for much more.

Weierstrass points. Let C be a smooth curve, p be a point of C, L be a line bundle on C, and $V \subset H^0(C, L)$ be a linear series of dimension $n+1$. It is not hard to see that $\#(\mathrm{ord}_p(V \smallsetminus \{0\})) = n+1$. Taking the integers in $\mathrm{ord}_p(V \smallsetminus \{0\})$ in increasing order, we obtain the sequence $a_0(p,V) < a_1(p,V) < \cdots < a_n(p,V)$, called the vanishing sequence of V at p. We will denote the vanishing sequence by $a(p,V)$ in the sequel. For a general point of C, the vanishing sequence at the point is $\{0, 1, \ldots, n\}$.

The shifted sequence $\alpha_0(p,V) \le \cdots \le \alpha_n(p,V)$ defined by $\alpha_i(p,V) := a_i(p,V) - i$ is called the ramification sequence of V at p. We will denote the ramification sequence by $\alpha(p,V)$. For a general point of C, the ramification sequence at the point is then $\{0, \ldots, 0\}$.

A point p of C is called a Weierstrass point (or inflectionary point) of V if $\alpha(p,V) \neq \{0, \ldots, 0\}$. The Weierstrass weight of V at p is $w(p,V) := \sum_{i=0}^{n} \alpha_i(p,V)$. A point p of C is a Weierstrass point of V if and only if $w(p,V) \neq 0$.

Suppose that C has genus g, and L has degree d. The Weierstrass weight obeys the following relation, called the Plücker formula:

$$\sum_{p \in C} w(p,V) = (n+1)d + \binom{n+1}{2}(2g-2). \tag{3.1}$$

Examples. (i) (Riemann-Hurwitz formula for maps to $\mathbb{P}^1$) Let V be a basepoint-free linear system of dimension $n+1 = 1+1$, so that V induces a map $\phi_{|V|}\colon C \to \mathbb{P}^1$. For each point p of C, let e_p denote the ramification index of $\phi_{|V|}$ at p. The vanishing sequence of V at p is equal to $\{0, e_p\}$, so that the ramification sequence is $\{0, e_p - 1\}$ and the Weierstrass weight at p is $e_p - 1$. The Plücker formula specializes to

$$\sum_{p \in C} (e_p - 1) = 2d + (2g - 2),$$

which, after rearrangement, is nothing but the Riemann-Hurwitz formula:

$$2g_C - 2 = d(2g_{\mathbb{P}^1} - 2) + \sum_{p \in C} (e_p - 1).$$

(ii) (Flex points on a smooth cubic) Suppose that C is smooth, and that V is a very ample linear system of dimension $n+1 = 2+1$ and degree 3 (so that $\phi_{|V|}$ embeds C as a smooth plane cubic).

Let p be a point of C. The order of vanishing of a section s in V at p is equal to the multiplicity $\mathrm{mult}_p(\ell_s.C)$ of intersection at p of the corresponding line ℓ_s in $\mathbb{P}^2$ with C. For a line ℓ not intersecting C at p, $\mathrm{mult}_p(\ell.C) = 0$; for a line intersecting C transversely at p, $\mathrm{mult}_p(\ell.C) = 1$; for a line tangent to C at p, $\mathrm{mult}_p(\ell.C) \in \{2, 3\}$.

If $\mathrm{mult}_p(T_pC.C) = 2$, then $a(p, V) = \{0, 1, 2\}$, hence $\alpha(p, V) = \{0, 0, 0\}$ and $w(p) = 0$. If $\mathrm{mult}_p(T_pC.C) = 3$ (i.e. p is a flex point of C), then $a(p, V) = \{0, 1, 3\}$, hence $\alpha(p, V) = \{0, 0, 1\}$ and $w(p) = 1$.

Plücker's formula then specializes to

$$\text{Number of flex points of } C = 3 \cdot 3 + \binom{3}{2}(2 \cdot 1 - 2) = 9,$$

recovering the count of nine flex points.

Weierstrass weight via principal parts bundles. The Weierstrass weight of V at p may be found by the following construction. As before, suppose that $\dim V = n+1$.

Taking top exterior powers of the morphism

$$\Psi\colon V \otimes_{\mathbb{C}} \mathcal{O}_C \to \mathcal{P}^n(L),$$

we obtain a morphism

$$\det\Psi\colon \bigwedge^{n+1} V \otimes \mathcal{O}_C \to \bigwedge^{n+1} \mathcal{P}^n(L).$$

Treating $\det\Psi$ as a global section of the locally free sheaf $\mathcal{H}om\left(\bigwedge^{n+1} V \otimes \mathcal{O}_C,\ \bigwedge^{n+1} \mathcal{P}^n(L)\right)$, we can ask about its order of vanishing at points of C.

Theorem ([EH16], p. 269)**.** *For each point p of C, the order of vanishing of the determinant $\det\Psi$ is equal to the Weierstrass weight $w(p)$ of V at p.*

3.5 ϕ-Weierstrass weight, and its behaviour in families

ϕ-Weierstrass weight. Let C be a curve (possibly singular), L be a line bundle on C, $V \subset H^0(C,L)$ be a linear series of dimension $n+1$, and $\phi\colon \Omega_C \to \omega_C$ be a derivation.

Define the ϕ-Weierstrass weight of V at p to be the order of vanishing[2] of the determinant of the morphism $V \otimes \mathcal{O}_C \to \mathcal{P}^n_\phi(L)$ at p (by analogy with the smooth case). We denote the ϕ-Weierstrass weight of V at p by $w_\phi(V,p)$ (or simply $w(V,p)$ or $w(p)$ if there is no danger of confusion).

When C is smooth and $\phi = \mathrm{id}$, the ϕ-Weierstrass weight is the usual Weierstrass weight (as described in the previous section).

In §3.6, we will describe a way to compute the ϕ-Weierstrass weight of a point by passing to the normalization of C.

[2]We recall that the order of vanishing of a section in the local ring $\mathcal{O}_{C,p} = A$ is defined to be $\mathrm{ord}_p(s) = \mathrm{length}_A(A/(s))$ [Ful98, p. 8]. When p is a regular point, ord_p becomes the usual discrete valuation of $\mathcal{O}_{C,p}$.

Varying $\phi\colon \Omega_C \to \omega_C$ in a family. The purpose of this paragraph is to make sense of an extension of the construction of §3.3 to the case of a family of curves.

Let S be a smooth surface, T be an scheme, $p\colon S\times T \to T$ be the projection of $S\times T$ onto the second factor, and $\mathcal{C}$ a Cartier divisor in $S\times T$ such that $p|_{\mathcal{C}}$ is flat.

Let f be a choice of local equation for $\mathcal{C}$ in an open subset U of $S\times T$. We have the following commutative diagram of sheaves over U where the left column is the exact sequence for $\Omega_{S\times T/T}$, and the rows depend on the choice of the local equation f:

$$\begin{tikzcd} p^*\Omega_T \arrow[d] & \\ \Omega_{S\times T} \arrow[r, "\wedge\frac{df}{f}"] \arrow[d] & \Omega^2_{S\times T}(\mathcal{C}) \arrow[d] \\ \Omega_{S\times T/T} \arrow[r, "\wedge\frac{df}{f}"'] & \Omega^2_{S\times T/T}(\mathcal{C}) \end{tikzcd} \qquad \text{over } U.$$

We thus obtain a morphism

$$\Omega_{S\times T} \to \Omega^2_{S\times T/T}(\mathcal{C}) \qquad \text{over } U. \tag{3.2}$$

Proceeding similarly to §3.3, we verify that after restricting to $\mathcal{C}$ the morphism (3.2) no longer depends on the local equation of $\mathcal{C}$. By gluing together local morphisms thus obtained, we construct a morphism $\phi_p\colon \Omega_{\mathcal{C}/T} \to \omega_{\mathcal{C}/T}$ on $\mathcal{C}$ that fits into the following diagram, where the top row is local and depends on the choice of f, whereas the bottom row is global (the diagram commutes for any choice of local equation for $\mathcal{C}$ over an open $U \subset S\times T$):

$$\begin{tikzcd} \Omega_{S\times T} \arrow[r] \arrow[d] & \Omega_{S\times T/T} \arrow[r, dashed, "\text{Over } U"] \arrow[d] & \Omega^2_{S\times T/T}(\mathcal{C}) \arrow[d, "\text{Restriction}"] \\ i_*\Omega_{\mathcal{C}} \arrow[r] & i_*\Omega_{\mathcal{C}/T} \arrow[r, "\exists", "i_*\phi_p"'] & i_*\omega_{\mathcal{C}/T} \end{tikzcd} \tag{3.3}$$

(where $i\colon \mathcal{C} \to S\times T$ is the inclusion map).

By composing the two maps in the bottom row of (3.3), we obtain a derivation $\phi\colon \Omega_{\mathcal{C}} \to$

$\omega_{C/T}$. For any $t \in T$, we obtain the following commutative diagram upon restricting to C_t:

$$\begin{tikzcd} \Omega_{C_t} \arrow[r, "\phi_t"] & \omega_{C_t} \\ \Omega_{\mathcal{C}}\big|_{C_t} \arrow[u] \arrow[ur] & \end{tikzcd}$$

where ϕ_t is the same derivation as the one constructed by the method of §3.3 for the curve $C_t \subset S$.

Let $\mathcal{E}$ be a vector bundle over $S \times T$, and $E_t := \mathcal{E}|_{C_t}$. By Theorem 2.25, we conclude that

$$\mathcal{P}^r_\phi(\mathcal{E})\big|_{C_t} \cong \mathcal{P}^r_{\phi_t}(E_t) \qquad \text{for any } r \geq 0,\ t \in T \tag{3.4}$$

The Weierstrass weight absorbed by a singular point in a degeneration. Continue with the set-up of the previous paragraph.

Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$, $\mathcal{V} \subset H^0(\mathcal{C}, \mathcal{L})$ be a linear system, $L_t = \mathcal{L}|_{C_t}$, and $V_t = \mathcal{V}|_{C_t}$.

Suppose that, for each $t \in T$, $|\mathcal{V}|$ does not contain any irreducible component of C_t (equivalently, that the restriction map $\mathcal{V} \to V_t$ is injective for all t).

Let

$$\mathcal{W} := V(\det(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{C}} \to \mathcal{P}^r_\phi(\mathcal{L})))$$

be the Weierstrass weight divisor.

By Theorem 2.25, we can conclude that

$$\det(\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{C}} \to \mathcal{P}^r_\phi(\mathcal{L}))\big|_{C_t} = \det(V_t \otimes_{\mathbb{C}} \mathcal{O}_{C_t} \to \mathcal{P}^r_{\phi_t}(L_t)).$$

Thus,

$$\mathcal{W} \cap C_t = \sum_{p \in C_t} w_{\phi_t}(p)\, p.$$

We obtain the following consequence.

Theorem 3.1. *Let T be an irreducible scheme, and $t_0 \in T$ a point. Suppose that the general* *fiber of $\mathcal{C} \to T$ is smooth, and p is a singular point of C_{t_0}. Then there exists a curve $D \subset T$, an open neighbourhood $t_0 \in U \subset D$, positive integers $w_1, \dots, w_k$ with $w_1 + \dots + w_k = w_{\phi_0}(p)$, and sections $p_1, \dots, p_k : U \to \mathcal{C}$, with $p_i(t)$ a Weierstrass point of C_t of weight w_i, and $p_i(0) = p$.*

Proof. The locus of T over which the fibre C_t is singular is closed, and similarly the locus of T over which some of the Weierstrass points meet is also closed. Therefore, restricting to a dense open subset U of T, we may assume that C_t is smooth and the cardinality of $\mathcal{W} \cap C_t$ independent of t. Then, it is sufficient to choose D to be a curve in T that passes through t_0 and with $D \cap U \neq \varnothing$. □

Theorem 3.1 gives a complete solution to the problem of computing how much Weierstrass weight gets absorbed by a singular point in a curve degeneration, which is the focus of this chapter.

3.6 Computing the ϕ-Weierstrass weight on the normalization of C

Although in principle the previous sections solve the problem of computing how much Weierstrass weight is absorbed by a singular point in a curve degeneration, the Weierstrass weight at a singular point may be difficult to compute from the definition.

In this section, we prove formula (3.10), which is a result of independent interest that is also useful for computing the ϕ-Weierstrass weight of a point in terms of quantities defined on the normalization of C.

As an intermediate step toward the proof of formula (3.10), we identify the vector bundle obtained by pulling back $\mathcal{P}^m_\phi(L)$ to the normalization of C. The statement (Theorem 3.5) is

$$\pi^* \mathcal{P}^m_\phi(L) \cong \mathcal{P}^m_{\cdot\Delta}(\pi^* L),$$

where $\pi : \tilde{C} \to C$ is the normalization of C, Δ is a divisor on $\tilde{C}$ that is constructed below, and $\cdot\Delta : K_{\tilde{C}} \to K_{\tilde{C}}(\Delta)$ is the morphism that is given by multiplying the sections of $K_{\tilde{C}}$ by the local equation of Δ.

The contents of this section are developed in the opposite order from the order of their introduction above — first, we construct the divisor Δ and compute Δ in two sequences of examples, then we identify the pullback of $\mathcal{P}^m_\phi(L)$ to the normalization of C, and then we prove formula 3.10 at the end of the section.

The divisor Δ and pullback of $\mathcal{P}^m_\phi(L)$ to the normalization of C. Continuing with the notation of the previous sections, let C be a curve contained in a smooth projective surface S, and $\pi\colon \tilde{C} \to C$ be the normalization of C.

We begin by constructing the divisor Δ.

Let $\operatorname{Sing} C$ denote the set of singular points of C, and let $\tilde{C}_0 \coloneqq \pi^{-1}(C \setminus \operatorname{Sing} C)$. Over $\tilde{C}_0$, the three line bundles $K_{\tilde{C}}|_{\tilde{C}_0}$, $\pi^*\omega_C|_{\tilde{C}_0}$ and $\pi^*\Omega_C|_{\tilde{C}_0}$ are all isomorphic. We have a commutative diagram

$$\begin{tikzcd} K_{\tilde{C}}(\tilde{C}_0) \arrow[r] & \pi^*\omega_C(\tilde{C}_0) \\ \pi^*\Omega_C(\tilde{C}_0) \arrow[u] \arrow[ur] & \end{tikzcd}$$

with all of the maps being isomorphisms.

Let U be an open neighbourhood in S in which C has local equation $f = 0$, and over which $\tilde{C}$ has a local coordinate t. Let $\tilde{U} = \pi^{-1}(U)$ and $\tilde{U}_0 = \pi^{-1}(U) \cap \tilde{C}_0$. Writing out the map $K_{\tilde{C}}(\tilde{C}_0) \to \pi^*\omega_C(\tilde{C}_0)$ in terms of the local frames induced by the choice of local coordinates,

$$dt \mapsto h\left(\frac{dx \wedge dy}{f}\right), \qquad h \in \mathcal{O}_{\tilde{C}}(U_0).$$

Write the normalization as $\pi\colon t \mapsto (\pi_1(t), \pi_2(t))$ in local coordinates in U. We chase the (class of) π^*dx in $\pi^*\Omega_C(U_0)$ through the diagram: going up to $K_{\tilde{C}}(U_0)$, π^*dx gets sent to $d\pi_1/dt\, dt$, which in turn gets sent to

$$\frac{d\pi_1}{dt}\, h\, \pi^*\left(\frac{dx \wedge dy}{f}\right) \qquad \text{in } \pi^*\omega_C(\tilde{U}_0).$$

On the other hand going diagonally, $\pi^* dx$ is sent to

$$\pi^*\left(dx \wedge \frac{\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy}{f}\right) = \pi^*\left(-\frac{\partial f}{\partial y}\frac{dx \wedge dy}{f}\right).$$

We conclude that

$$\frac{d\pi_1}{dt}\,h = -\pi^*\frac{df}{dy} \qquad \text{in } \mathcal{O}_{\tilde{C}}(\tilde{U}_0) \tag{3.5}$$

The section h extends to a possibly meromorphic section over $\tilde{U}$, which we again denote by h. We would like to show that the extension of h over $\tilde{U}$ is in fact holomorphic, or equivalently that $\mathrm{ord}_r(h) \geq 0$ for all $r \in \tilde{U}$. From equality (3.5), we see that

$$\mathrm{ord}_r(h) = \mathrm{ord}_r\left(\pi^*\frac{\partial f}{\partial y}\right) - \mathrm{ord}_r\left(\pi^* dx\right) = -\,\mathrm{ord}_r\ \pi^*\left(\frac{dx}{\partial f/\partial y}\right). \tag{3.6}$$

Therefore, it is enough to show that $\mathrm{ord}_r(\pi^* dx) \leq \mathrm{ord}_r\left(\pi^*\frac{\partial f}{\partial y}\right)$.

Let $r \in \tilde{C}$. Choose an open neighbourhood U of $\pi(r)$ in S, with local coordinates x, y, in which C is given by the local equation $f = 0$, where f has a power series expansion

$$f(x,y) = \sum c_{ij} x^i y^j,$$

and such that the branch of the normalization $\tilde{C}$ that contains the point r has a local coordinate t. By means of translations, the coordinates may be chosen so that r has coordinate $t = 0$, and $\pi(r)$ has coordinates $(x,y) = (0,0)$. Finally, after possibly linearly changing coordinates on C, the coordinates may be chosen so that neither $\pi^* x$ nor $\pi^* y$ is identically zero on $\tilde{C}$ (a general linear change of coordinates will achieve this).

Lemma 3.2. *With the setup of the previous paragraph, we have* $\mathrm{ord}_r\left(\pi^* dx\right) \leq \mathrm{ord}_r\left(\pi^*\frac{\partial f}{\partial y}\right)$.

Proof. Set

$$a := \operatorname{ord}_r(\pi^* x), \qquad b := \operatorname{ord}_r(\pi^* y),$$
$$d := \min\{ai + bj : c_{ij} \neq 0\}, \qquad S_d = \{(i,j) : c_{ij} \neq 0,\ ai + bj = d\}.$$

We observe that

(i) $a \geq 1$ and $b \geq 1$, since $r \in \pi^{-1}((0,0))$ by assumption.

(ii) $|S_d| \geq 2$, since $\pi^* f = 0$ and neither $\pi^* x$ nor $\pi^* y$ is identically zero by the choice of coordinate system (there must be at least two monomials of lowest degree to have cancellation).

(iii) Therefore, there is a point $(i,j) \in S_d$ with $j \neq 0$.

We have

$$\operatorname{ord}_r(\pi^* dx) = a - 1,$$

and

$$\frac{\partial f}{\partial y} = \sum j\, c_{ij} x^i y^{j-1}, \qquad \text{so that} \qquad \operatorname{ord}_r\left(\pi^* \frac{\partial f}{\partial y}\right) = d - b,$$

The statement we want to show is

$$a - 1 \leq d - b \qquad \text{or, equivalently and symmetrically,} \qquad d \geq a + b - 1.$$

We achieve this by considering several cases:

First, suppose that there exists $(i,j) \in S$ with $i \neq 0$ and $j \neq 0$. Then the statement is clear because $a \geq 1$ and $b \geq 1$:

$$d = ai + bj \geq a + b > a + b - 1$$

Otherwise, for all $(i,j) \in S_d$, we have $i \cdot j = 0$. Then $S_d = \{(i_d, 0),\ (0, j_d)\}$, where $d = ai_d =$

bj_d (as argued previously, S_d has at least two elements).

If $i_d \geq 2$ and $j_d \geq 2$, then $a \leq d/2$ and $b \leq d/2$, so that $d \geq a + b > a + b - 1$. Otherwise, at least one of i_d and j_d is equal to 1.

Suppose that $i_d = 1$. Then $c_{10} \neq 0$, C is smooth at $(0,0)$, and π is a local isomorphism. The tangent line to C at $(0,0)$ is given by the equation $c_{10}x + c_{01}y = 0$. Since $c_{10} \neq 0$, the line $y = 0$ is not tangent. Therefore,

$$b = \operatorname{ord}_r(\pi^* y) = 1 \qquad \text{and so} \qquad d = a \cdot i_d = a \cdot 1 = a + 1 - 1 = a + b - 1.$$

Similarly, if $j_d = 1$, then $x = 0$ is not tangent to C at $(0,0)$, hence $a = \operatorname{ord}_r(\pi^* x) = 1$ and $d = a + b - 1$. $\square$

Given an open cover $\{U_i\}$ of C in S, it is checked immediately that the sections h_i in $\mathcal{O}_{\tilde{C}}(\tilde{U}_i)$ glue to a section in $\mathcal{O}_{\tilde{C}}(\tilde{C})$, which we again call h. Let Δ be the divisor of zeros of h (since h was shown to be holomorphic, Δ is an effective divisor).

We now introduce some notation and language for working with Δ. Let $\operatorname{Sing} C$ denote the set of singular points of C, let $p \in \operatorname{Sing} C$ be a singular point, suppose that C is given by the local equation $f = 0$ in a neighbourhood of p, and let $\pi : \tilde{C} \to C$ denote the normalization of C.

We set $\pi^{-1}(p) =: \{r_1, \ldots, r_k\}$, and, for each $i = 1, \ldots, k$, we set

$$\gamma_i := -\operatorname{ord}_{r_i} \pi^* \left(\frac{dx}{\partial f / \partial y(x,y)} \right) = -\operatorname{ord}_{r_i} \pi^* \left(\frac{dy}{\partial f / \partial x(x,y)} \right).^{3}$$

The adjoint divisor Δ_p of p is then defined as

$$\Delta_p := \sum_{i=1}^{k} \gamma_i r_i$$

[3]One of $\partial f/\partial x$ or $\partial f/\partial y$ could vanish along C; however (in characteristic 0), the restriction of at least one of the two to C will be generically nonzero. Alternatively, both partial derivatives will be generically nonzero after a general linear change of coordinates.

and equality (3.6) shows that

$$\Delta = \sum_{p \in \operatorname{Sing} C} \Delta_p.$$

Following tradition, we call the divisor Δ the adjoint divisor of the normalization morphism $\pi : \tilde{C} \to C$ — indeed, this is the name under which the divisor Δ appears in the classical literature on algebraic curves. One of the main reasons Δ is studied classically is the result of Theorem 3.3 below — Δ allows us to study the complete linear series $|\omega_C|$ on C by instead considering the complete canonical linear series of $\tilde{C}$ twisted by the adjoint divisor $|K_{\tilde{C}}(\Delta)|$ on the normalization $\tilde{C}$ of C. Theorem 3.3 will also be useful below for identifying the pullback of $\mathcal{P}^m_\phi(L)$ to the normalization of C. For more information about Δ, we kindly refer the reader to [ACGH84, App. A].

Theorem 3.3. $\pi^*\omega_C \cong K_{\tilde{C}}(\Delta)$.

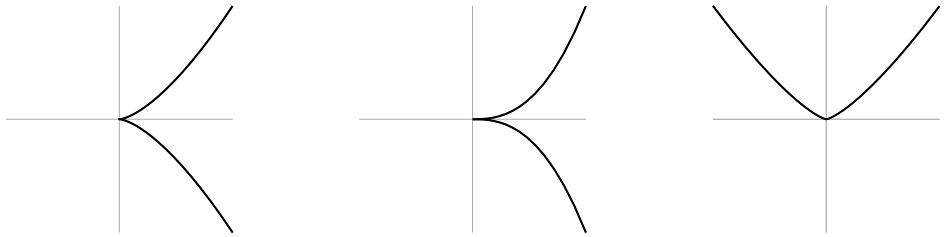
Proof. This follows from the construction of Δ. □

Next, we compute Δ in two families of examples.

Example 3.4. (i) Let m, n be coprime positive integers, and let $C_{m,n} \subset \mathbb{P}^2$ be the curve with the equation $Y^m Z^n - X^n Z^m = 0$.

The case $m = 2, n = 3$ is a cuspidal cubic; more generally the curve in the case $m = 2$ and n arbitrary has an n-th order cusp.

The curve $C_{m,n}$ has a single singularity of multiplicity $\min(m,n)$ at $p = [0 : 0 : 1]$. In the affine chart $Z \neq 0$, C is given by the local equation $y^m - x^n = 0$. The normalization $\pi : \tilde{C}_{m,n} \to C_{m,n}$ has a single branch and locally analytically around the singularity may be given by the map $t \mapsto (t^m, t^n)$. Let $r = \pi^{-1}(p)$.



$m = 2,\ n = 3$ $m = 2,\ n = 5$ $m = 3,\ n = 4$

Figure 3.1: The (real points of) three of the curves $C_{m,n}$ in the chart $Z \neq 0$ of $\mathbb{P}^2$. The values of m and n given below each figure indicate which curve $C_{m,n}$ is plotted.

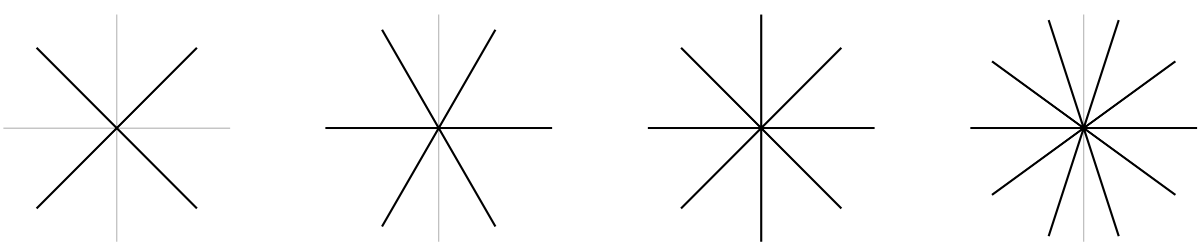
We have

$$\pi^*\left(\frac{dx}{\partial f/\partial y}\right) = \frac{d(t^m)}{p(t^n)^{m-1}} = \frac{m\,t^{m-1}\,dt}{m\,t^{n(m-1)}} = \frac{dt}{t^{(m-1)(n-1)}},$$

so that the adjoint divisor is $\Delta = (m-1)(n-1)r$, with $\deg \Delta = (m-1)(n-1)$.

In the special case of the cuspidal cubic, $\Delta = 2r$; more generally, for an n-th order cusp (the case $m = 2,\ n$ arbitrary), $\Delta = (n-1)r$.

(ii) (Ordinary n-tuple point) Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be constants such that $a_i b_j \neq a_j b_i$ for all $i \neq j$, and let C_n be the curve with equation $(a_1X + b_1Y)(a_2X + b_2Y)\cdots(a_iX + b_iY) = 0$ (the condition on the coefficients is satisfied exactly when the lines $a_iX + b_iY = 0$ are pairwise distinct). The case $n = 2$ is a node.



$n = 2$ $n = 3$ $n = 4$ $n = 5$

Figure 3.2: The (real points of) four of the curves C_n in the chart $Z \neq 0$ in $\mathbb{P}^2$. The values of n given below each figure indicate which curve C_n is plotted.

The curve C_n has a single singularity of multiplicity n at the point $p = [0:0:1]$. Passing to the affine chart $Z \neq 0$, the equation of C_n is

$$f(x,y) = p_n(a,b)x^n + p_{n-1}(a,b)x^{n-1}y + \cdots + p_0(a,b)y^n,$$

where $p_n(a,b)$ is a homogeneous polynomial in $a_1,\ldots,a_n,b_1,\ldots,b_n$ of degree n. Then

$$\partial f/\partial y = p_{n-1}(a,b)x^{n-1} + 2p_{n-2}(a,b)x^{n-2}y + \cdots + n\,p_0(a,b)y^{n-1}.$$

The normalization $\pi\colon \widetilde{C}_n \to C_n$ has n branches, which locally analytically can be given the form $\pi_i\colon t \mapsto (b_i t,\ -a_i t),\ i = 1,\ldots,n$. Set $r_i := \pi_i^{-1}(p)$. We have

$$\pi_i^*(\partial f/\partial y) = P(a,b)t^{n-1}$$

where $P(a,b)$ is a polynomial in $a_1,\ldots,b_n$ ($P(a,b) \neq 0$ for a general choice of a_is and b_js), so that $\pi_i^*(\frac{dx}{\partial f/\partial y}) = a_i\,dt/(P(a,b)t^{n-1})$; the latter has a pole of order $n-1$ at r_i.

We conclude that $\Delta = (n-1)(r_1 + \cdots + r_n)$, and $\deg\Delta = n(n-1)$.

In particular, for a node ($n = 2$), $\Delta = r_1 + r_2$.

Finally, we identify the bundle obtained by pulling back $\mathcal{P}^m_\phi(L)$ to the normalization of C. Let $\cdot\Delta\colon K_{\tilde{C}} \to K_{\tilde{C}}(\Delta)$ denote the morphism that is given by multiplying the sections of $K_{\tilde{C}}$ by the local equation of Δ.

Theorem 3.5. *For any line bundle L on C, and any integer $m \geq 0$,*

$$\pi^*\mathcal{P}^m_\phi(L) \cong \mathcal{P}^m_{\cdot\Delta}(\pi^*L).$$

Proof. By the construction of Δ and Theorem 3.3, the diagram

$$\begin{tikzcd} K_{\tilde{C}} \arrow[r, "\cdot\Delta"] & \pi^*\omega_C = K_{\tilde{C}}(\Delta) \\ \pi^*\Omega_C \arrow[u, "d\pi"] \arrow[ur, "\pi^*\phi"'] & \end{tikzcd} \tag{3.7}$$

commutes.

The claimed isomorphism now follows by functoriality (Theorem 2.24). $\square$

Computing the ϕ-Weierstrass weight on $\tilde{C}$. Continuing with the notation of the previous section, let $V \subset H^0(C, L)$ be a linear series of dimension $n+1$ on a curve $C \subset S$. From now on, we suppose that none of the sections of V vanish along an irreducible component of C. Let $\pi : \tilde{C} \to C$ be the normalization.

The pullback morphism $\pi^* L \to \pi^* \mathcal{P}^m_\phi(L) = \mathcal{P}^m_{\cdot\Delta}(\pi^* L)$ is an order m differential operator, so by the universal property of the m-th principal parts bundle on $\tilde{C}$, we have a commuting triangle of sheaves of abelian groups

$$\begin{tikzcd} \pi^* L \arrow[r] \arrow[dr] & \mathcal{P}^m(\pi^* L) \arrow[d] \\ & \mathcal{P}^m_{\cdot\Delta}(\pi^* L) \end{tikzcd}$$

which gives rise to the commuting triangle of $\mathcal{O}_{\tilde{C}}$-modules

$$\begin{tikzcd} \pi^* V \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{C}} \arrow[r] \arrow[dr] & \mathcal{P}^m(\pi^* L) \arrow[d] \\ & \mathcal{P}^m_{\cdot\Delta}(\pi^* L) \end{tikzcd}$$

where $\pi^* V = \{\pi^* f : f \in V\} \subset H^0(\tilde{C}, \pi^* L)$.

Taking top exterior powers, we obtain a commuting triangle of line bundles

$$\begin{tikzcd} \bigwedge^{m+1} \pi^* V \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{C}} \arrow[r] \arrow[dr] & \bigwedge^{m+1} \mathcal{P}^m(\pi^* L) \arrow[d] \\ & \bigwedge^{m+1} \mathcal{P}^m_{\cdot\Delta}(\pi^* L) \end{tikzcd} \tag{3.8}$$

Lemma 3.6. *Let C be a smooth curve, q be a point of C, L a line bundle on C, $d \geq 0$ and $m \geq 1$ be integers, $D = dq$ be a divisor on C, and $\Phi_{m,D}$ be the map $\Phi_{m,D} : \mathcal{P}^m(L) \to \mathcal{P}^m_{\cdot D}(L)$ (here $\mathcal{P}^m_{\cdot D}(L)$ is the principal parts bundle corresponding to the derivation $\Omega_C \xrightarrow{\cdot D} \Omega_C(D)$).*

Then

$$\operatorname{ord}_p\left(\det \Phi_{m,D}\right) = \begin{cases} d\binom{m+1}{2}, & p = q, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By [Har77], Ex. II.V.16(d), for any short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of locally free sheaves, of ranks n', n and n'', respectively, there is an isomorphism of exterior powers

$$\bigwedge^{n} \mathcal{F} \cong \bigwedge^{n'} \mathcal{F}' \otimes \bigwedge^{n''} \mathcal{F}''. \tag{3.9}$$

Moreover, the isomorphism (3.9) is functorial in the sense that a morphism of short exact sequences of locally free sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\ 0 & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}'' & \longrightarrow & 0 \end{array}$$

where $\mathcal{G}', \mathcal{G}$ and $\mathcal{G}''$ have ranks n', n and n'', respectively, yields a commuting diagram

$$\begin{array}{ccc} \bigwedge^n \mathcal{F} & \xrightarrow{\cong} & \bigwedge^{n'} \mathcal{F}' \otimes \bigwedge^{n''} \mathcal{F}'' \\ \downarrow{\scriptstyle \wedge^n \phi} & & \downarrow{\scriptstyle \wedge^{n'} \phi' \otimes \wedge^{n''} \phi''} \\ \bigwedge^n \mathcal{G} & \xrightarrow{\cong} & \bigwedge^{n'} \mathcal{G}' \otimes \bigwedge^{n''} \mathcal{G}'' \end{array}$$

Therefore, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_C^{\otimes m} & \longrightarrow & \mathcal{P}^m & \longrightarrow & \mathcal{P}^{m-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Phi_{m,D}} & & \downarrow{\scriptstyle \Phi_{m-1,D}} & & \\ 0 & \longrightarrow & \Omega_C(D)^{\otimes m} & \longrightarrow & \mathcal{P}^m_{\cdot D} & \longrightarrow & \mathcal{P}^{m-1}_{\cdot D} & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & \Omega_C^{\otimes m}(mD) & & & & & & \end{array}$$

yields the diagram

$$\begin{tikzcd}
\bigwedge^{m+1} \mathcal{P}^m \arrow[r, "\cong"] \arrow[d, "\wedge^{m+1}\Phi_{m,D}"] & \bigwedge^m \mathcal{P}^{m-1} \otimes \Omega_C^{\otimes m} \arrow[d, "\wedge^m \Phi_{m-1,D} \otimes (\cdot mD)"] \\
\bigwedge^{m+1} \mathcal{P}^m_{\cdot D} \arrow[r, "\cong"] & \bigwedge^m \mathcal{P}^{m-1}_{\cdot D} \otimes \Omega^{\otimes m}(mD)
\end{tikzcd}$$

Recursing, we obtain

$$\begin{tikzcd}
\bigwedge^{m+1} \mathcal{P}^m \arrow[r, "\cong"] \arrow[d, "\det \Phi_{m,D}"] & \mathcal{O}_C \otimes \Omega_C \otimes \Omega_C^{\otimes 2} \otimes \cdots \otimes \Omega_C^{\otimes m} \arrow[d, "\mathrm{id} \otimes (\cdot D) \otimes (\cdot 2D) \otimes \cdots \otimes (\cdot mD)"] \\
\bigwedge^{m+1} \mathcal{P}^m_{\cdot D} \arrow[r, "\cong"] & \mathcal{O}_C \otimes \Omega_C(D) \otimes \Omega_C^{\otimes 2}(2D) \otimes \cdots \otimes \Omega_C^{\otimes m}(mD)
\end{tikzcd}$$

We see that $\det \Phi_{m,D}$ has order of vanishing $d(1 + 2 + \cdots + m) = d\binom{m+1}{2}$ at p, and does not vanish outside of p. $\square$

We will also need the following simple consequence of the push-pull formula.

Lemma 3.7. *Let $\pi : \tilde{C} \to C$ be the normalization of C, L be a line bundle on C of degree d, s be an section in $H^0(C, L)$, p be a point of C, and $\pi^{-1}(p) = \{r_1, \ldots, r_m\}$ be the fibre of π over p. Then*

$$\operatorname{ord}_p(s) = \operatorname{ord}_{r_1}(\pi^* s) + \cdots + \operatorname{ord}_{r_m}(\pi^* s).$$

Proof. Let $[s]$ be the divisor corresponding to the section s. By the push-pull formula ([Ful98, Proposition 2.3(c)]), we have

$$\pi_*(\pi^* [s] \cap \tilde{C}) = [s] \cap \pi_* \tilde{C}.$$

Since π is a degree one map, $\pi_* \tilde{C} = C$, and so

$$\pi_*\Big(\sum_{r \in \tilde{C}} \operatorname{ord}_r(\pi^* s)\, r\Big) = \sum_{x \in C} \operatorname{ord}_x(s)\, x.$$

Comparing the coefficient of the point class p on both sides of the equality, we obtain the claim. $\square$

For any point $r \in \tilde{C}$, let $\tilde{w}(r)$ denote the weight of the linear series $\pi^* V = \{\pi^* f : f \in V\} \subset H^0(\tilde{C}, \pi^* L)$ at r. We then have the following formula.

Theorem 3.8. *For each point p of C,*

$$w_\phi(p) = (\deg \Delta_p)\binom{n+1}{2} + \sum_{r \in \pi^{-1}(p)} \tilde{w}(r). \tag{3.10}$$

Proof. By definition, $w_\phi(p) = \operatorname{ord}_p \det(V \otimes_{\mathbb{C}} \mathcal{O}_C \to \mathcal{P}^n_\phi(L))$. By Lemma 3.7, the fact that $\pi^* \mathcal{P}^n_\phi(L) \cong \mathcal{P}^n_{\cdot\Delta}(\pi^* L)$ (Theorem 3.5), and functoriality (Theorem 2.24), the latter is equal to the sum

$$\sum_{r \in \pi^{-1}(p)} \operatorname{ord}_r \pi^* \det(V \otimes_{\mathbb{C}} \mathcal{O}_C \to \mathcal{P}^n_\phi(L)) = \sum_{r \in \pi^{-1}(p)} \operatorname{ord}_r \det(\pi^* V \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{C}} \to \mathcal{P}^n_{\cdot\Delta}(\pi^* L)).$$

We recall that $\Delta_p = \sum_{r \in \pi^{-1}(p)} \gamma_r r$. Now, for $r \in \pi^{-1}(p)$, by commutativity of diagram (3.8) and Lemma 3.6,

$$\begin{aligned} \operatorname{ord}_r \det(\pi^* V \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{C}} \to \mathcal{P}^n_{\cdot\Delta}(\pi^* L)) &= \operatorname{ord}_r \det(\pi^* V \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{C}} \to \mathcal{P}^n(\pi^* L)) \\ &\quad + \operatorname{ord}_r \det(\mathcal{P}^n(\pi^* L) \to \mathcal{P}^n_{\cdot\Delta}(\pi^* L)) \\ &= \tilde{w}(r) + \gamma_r \binom{n+1}{2}. \end{aligned}$$

Taking the sum of the above terms over all $r \in \pi^{-1}(p)$, we obtain the statement of the theorem. □

3.7 Examples

(i) (Cubic cusp) Let $C \subset \mathbb{P}^2$ be the curve cut out by the equation $Y^2 Z - X^3 = 0$. As is well-known, C has a single singularity of multiplicity two at the point $p = [0:0:1]$.

We pass to the affine chart $U = \{[X:Y:Z] \in \mathbb{P}^3 : Z \neq 0\}$ with coordinates $x = X/Z$, $y = Y/Z$ (in which C is cut out by the equation $y^2 - x^3 = 0$, and the cusp has coordinates $(0,0)$).

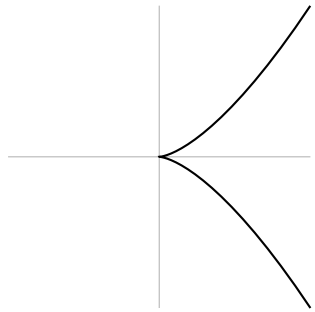


Figure 3.3: The (real points of the) cubic cusp $Y^2Z = X^3$ in the chart $Z \neq 0$ of $\mathbb{P}^2$.

Let $L = i^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_C(3)$ and $V = H^0(C, L)$; $h^0(C, L) = 3$. We apply formula (3.10) to compute $w(p)$ at the cusp.

Let $\pi\colon \tilde{C} \to C$ denote the normalization map. There are local analytic neighbourhoods $p \in V$ and $\pi^{-1}(p) \subset \tilde{V}$ (the latter with coordinate t) in which π has the form $t \mapsto (t^2, t^3) \subset U$. The cusp p has a single preimage (with coordinate $t = 0$), which we denote by r. The linear series π^*V is spanned by

$$\pi^*x = t^2, \quad \pi^*y = t^3, \quad \pi^*1 = 1,$$

and

$$\mathrm{ord}_r(t^2) = 2, \quad \mathrm{ord}_r(t^3) = 3, \quad \mathrm{ord}_r(1) = 0,$$

so that the vanishing and ramification sequences of π^*V at r are

$$a(\pi^*V, r) = \{0, 2, 3\},$$
$$\alpha(\pi^*V, r) = \{0, 1, 1\},$$

respectively, and $\tilde{w}(r) = 2$. As computed in example 3.4(i), the adjoint divisor of the normalization is $2r$, so has degree 2, and we conclude that

$$w(p) = \deg\Delta \cdot \binom{3}{2} + \sum_{r\in\pi^{-1}(p)} \tilde{w}(r) = 2\cdot 3 + 2 = 8.$$

Applying Proposition 3.1 to the family of elliptic curves degenerating to a cuspidal cubic, the above computation recovers the count of 8 flex points absorbed into the cusp point.

(ii) (Cubic node) Suppose that C is given by the equation $Y^2Z - X^3 - X^2Z = 0$ in $\mathbb{P}^2$. It is again likely known to the reader that C has an ordinary double point at $p = [0:0:1]$ and no other singularities.

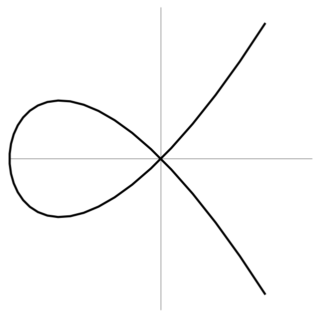


Figure 3.4: The (real points of the) cubic cusp $Y^2Z = X^3 + X^2Z$ in the chart $Z \neq 0$ of $\mathbb{P}^2$.

In the affine chart $U = \{Z \neq 0\} \subset \mathbb{P}^2$, C is cut out by the equation $y^2 = x^3 + x^2$, and the node has coordinates $(0,0)$.

Let $L = i^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_C(3)$ and $V = H^0(C, L)$. We again apply formula 3.10 to compute the Weierstrass weight of V at p.

Let $\pi\colon \tilde{C} \to C$ denote the normalization map. We can choose local analytic coordinates in which π has the form $t \mapsto (t^2-1, t(t^2-1)) \in U$. The preimage of p consists of two points with coordinates $t=-1$ and $t=1$, which we call r_- and r_+, respectively.

We have

$$\pi^*x = t^2 - 1 = (t-1)(t+1), \quad \pi^*y = t(t^2-1) = (t-1)t(t+1), \quad \pi^*1 = 1.$$

For the purpose of computing the vanishing and ramification sequences, a better-suited basis of π^*V is

$$\pi^*(y+x) = (t-1)(t+1)^2, \quad \pi^*(y-x) = (t-1)^2(t+1), \quad \pi^*1 = 1$$

(the first two sections are of course equations of the two tangent lines). The orders of vanishing of the latter at r_- and r_+ are

$$\begin{array}{c|c} \operatorname{ord}_-(\pi^*(y+x)) = 1 & \operatorname{ord}_+(\pi^*(y+x)) = 2 \\ \operatorname{ord}_-(\pi^*(y-x)) = 2 & \operatorname{ord}_+(\pi^*(y-x)) = 1 \\ \operatorname{ord}_- 1 = 0 & \operatorname{ord}_+ 1 = 0 \end{array}$$

respectively.

The vanishing and ramification sequences are

$$\begin{array}{c|c} a(\pi^*V, r_-) = \{0,1,2\} & a(\pi^*V, r_+) = \{0,1,2\} \\ \alpha(\pi^*V, r_-) = \{0,0,0\} & \alpha(\pi^*V, r_+) = \{0,0,0\} \end{array}$$

and so $\tilde{w}(r_-) = \tilde{w}(r_+) = 0$.

In Example 3.4(ii), we computed that $\Delta = r_- + r_+$, so $\deg\Delta = 2$. It follows by formula (3.10) that

$$w(p) = 2 \cdot \binom{3}{2} + 0 = 6.$$

Applying Proposition 3.1 to a family of elliptic curves degenerating to a nodal cubic, the above computation recovers the count of 6 flex points absorbed into the nodal point.

(iii) (n-th order node) Let $C_n \subset \mathbb{P}^2$ be the curve with equation $Y^2Z^{2n-2} - X^{2n} = 0$ (locally analytically, the case $n = 1$ is the same as the nodal cubic of the previous example) and take $L = i^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $V = H^0(C_n, L)$. We pass to the chart $Z \neq 0$, where the local equation is $y^2 - x^{2n} = 0$.

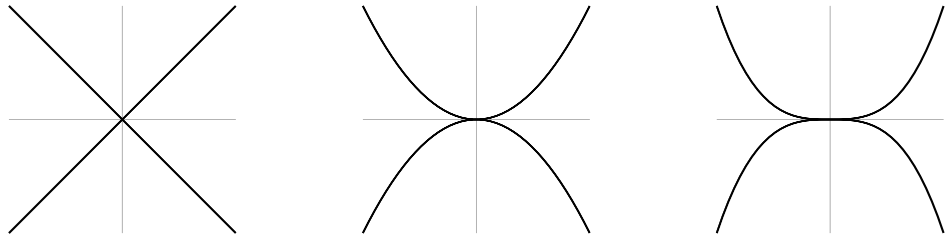


Figure 3.5: The (real points of) three of the curves C_n in the chart $Z \neq 0$ of $\mathbb{P}^2$. The values n given below each figure indicate which curve C_n is plotted.

The normalization of C_n has two branches, one of which is given locally analytically by $\pi_1 : t \to (t, t^n)$ and the other by $\pi_2 : t \to (t, -t^n)$. Let $r_1 = \pi_1^{-1}(p)$ and $r_2 = \pi_2^{-1}(p)$. We find that $\partial f / \partial y = 2y$, so that

$$\pi_1^* \left(\frac{dx}{\partial f / \partial y} \right) = \frac{dt}{2t^n}, \qquad \pi_2^* \left(\frac{dx}{\partial f / \partial y} \right) = \frac{dt}{-2t^n}$$

and so

$$\Delta = n(r_1 + r_2).$$

In the case $n = 1$, both of the components of the nodal curve are contained in $|\mathcal{O}_{\mathbb{P}^2}(1)|$, so formula (3.10) does not apply. So suppose that $n \geq 2$. We have

$\pi_1^* x = t$	$\pi_2^* x = t$
$\pi_1^* y = t^n$	$\pi_2^* y = -t^n$
$\pi_1^* 1 = 1$	$\pi_2^* 1 = 1$

so that

$a(\pi_1^* V, r_1) = \{0, 1, n\}$	$a(\pi_2^* V, r_2) = \{0, 1, n\}$
$\alpha(\pi_1^* V, r_1) = \{0, 0, n-2\}$	$a(\pi_2^* V, r_2) = \{0, 0, n-2\}$

and so $\tilde{w}(r_1) = \tilde{w}(r_2) = n - 2$. Applying formula (3.10), we see that

$$w(p) = \deg \Delta \binom{3}{2} + 2(n-2) = 8n - 4.$$

(iv) Let p, q be coprime positive integers and let $C_{p,q}$ be the curve with the equation $Y^q Z^p -$

$X^pZ^q = 0$ in $\mathbb{P}^2$ (the case $p = 2, q = 3$ is the cuspidal cubic), and once again choose $L = i^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $V = H^0(C_{p,q}, L)$. We pass to the affine chart $Z \neq 0$, in which the curve has the equation $y^p - x^q = 0$. The normalization has a single branch, and locally analytically can be given the form $t \mapsto (t^p, t^q)$. Set $r = \pi^{-1}(p)$ (p is the singular point of $C_{p,q}$ and $\pi\colon \tilde{C}_{p,q} \to C_{p,q}$ is the normalization map). We have

$$\pi^*x = t^p, \quad \pi^*y = t^q, \quad \pi^*1 = 1,$$

which have orders of vanishing

$$\operatorname{ord}_r(\pi^*x) = p, \quad \operatorname{ord}_r(\pi^*y) = q, \quad \operatorname{ord}_r 1 = 0.$$

The vanishing and ramification sequences are then

$$a(\pi^*V, r) = \{0, \min(p,q), \max(p,q)\}$$
$$\alpha(\pi^*V, r) = \{0, \min(p,q) - 1, \max(p,q) - 2\},$$

so that $\tilde{w}(r) = p + q - 3$, and

$$w(p) = (p-1)(q-1)\binom{3}{2} + p + q - 3 = 3(p-1)(q-1) + p + q - 3 = 3pq - 2p - 2q.$$

To give an example with a different line bundle, suppose that $V' = H^0(C_{p,q}, i^*\mathcal{O}_{\mathbb{P}^2}(2))$ (say). We have

$$\pi^*x^2 = t^{2p}, \quad \pi^*(xy) = t^{p+q}, \quad \pi^*y^2 = t^{2q}, \quad \pi^*x = t^p, \quad \pi^*y = t^q, \quad \pi^*1 = 1.$$

Suppose that $p < q$ (there is no loss of generality, as this can be achieved by a change of coordinates). Then the vanishing and ramification sequences are

$$a(\pi^* V', r) = \{0,\, p,\, q,\, 2p,\, p+q,\, 2q\}$$
$$\alpha(\pi^* V', r) = \{0,\, p-1,\, q-2,\, 2p-3,\, p+q-4,\, 2q-5\}$$

Then $\tilde{w}(r) = 4p + 4q - 15$ and

$$w(p) = 3pq + p + q - 12.$$

Chapter 4

Applications, part II: Counting lines in a linear system in a Grassmannian

One of the most beautiful facts of classical algebraic geometry is that every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines. In this chapter, as an application of ϕ-principal-part-bundles, we describe a method for computing the number of lines in a linear system in a variety $X \subset \mathbb{P}^N$ (in the case when the number of lines is finite, conditional on a certain degeneracy locus having the expected dimension). Applying this method, we find explicit expressions for numbers of lines that appear in a linear system (of degree and dimension such that the expected number of lines is finite) in $\mathbb{G}(k,n)$ (including the special cases $k = 0$ and $k = n - 1$ when the Grassmannian is isomorphic to a projective space); the count of 27 lines in a smooth cubic is recovered as a special case.

We now summarize the contents of this chapter.

In §4.1, we set down preliminary facts on Fano varieties of k-planes, Grassmannians, projective bundles, and the hooklength formula. Facts from these sections will be used throughout the chapter, sometimes without explicit mention of them. This section also serves to set down our notation.

In §4.2, we describe our ϕ-principal-parts-bundle method for computing the number of lines that appear in a linear system in a variety $X \subset \mathbb{P}^N$. We kindly refer the reader to

Chapter 1 for a sketch of this method. The method translates the computation of the number of lines that appear in a linear system $|V| \subset |L|$ of degree d and projective dimension $d - \dim(F_1(X)) + 1$ (where $F_1(X)$ denotes the Fano variety of lines in X) to the computation of the Chern class

$$\pi_{X,*}\ c_{\dim F_1(X)}(\mathcal{P}^d_\phi(\pi_X^* L)).$$

This result is stated as Theorem 4.3, and concludes §4.2.

In §4.3, we apply the method to compute the number of lines in a linear system of degree d and projective dimension $d - 2n + 3$ in projective space. The result is

$$\sum_{k=n-1}^{2n-2} (-1)^{k-n+1}\, \sigma(d+1,\, d+1-k) \binom{d+1-k}{2n-2-k} \binom{k+1}{n} d^{2n-2-k}.$$

Together with several hypotheses that are necessary for our treatment, the result is stated as Theorem 4.4. Several examples are computed following the statement of Theorem 4.4, including the count of 27 lines in a cubic surface in $\mathbb{P}^3$ and of 320 lines in a pencil of K3 surfaces in $\mathbb{P}^3$.

In §4.4, we apply the method to compute the number of lines in a linear system in $\mathbb{G}(k,n)$ (in its Plücker embedding). We treat the cases $1 \le k \le n-2$ in this section, excluding the cases $k = 1$ and $k = n - 1$; the latter two cases require a slight modification of the methods used in §4.4 (and covered by the previous section, as in these cases the two Grassmannians are isomorphic to projective spaces).

The computation in the case of a Grassmannian requires some preliminary work.

In §4.4.1, we compute the number of lines in a quartic surface in $\mathbb{G}(1,3)$ by a different method, as a test case for the general computation.

In §4.4.2, we describe the Picard group of a flag variety of type $\mathbb{F}l(k, k+1, \ldots, k+m; n)$. The motivation for this is that the universal family $\mathcal{U}_1(\mathbb{G}(k,n))$ over the Fano variety $F_1(\mathbb{G}(k,n))$ of lines in $\mathbb{G}(k,n)$ can be identified with the flag variety $\mathbb{F}l(k-1, k, k+1; n)$, and facts about the Picard group of $\mathcal{U}_1(\mathbb{G}(k,n))$ will be useful for the computation of the number of lines.

The identifications $\mathcal{U}_1(\mathbb{G}(k,n)) = \mathbb{F}l(k-1,k,k+1;n)$ and $F_1(\mathbb{G}(k,n)) = \mathbb{F}l(k-1,k+1;n)$ are made in §4.4.3. We also compute the dimension of $F_1(\mathbb{G}(k,n))$ and establish that it is irreducible.

In §4.4.4, we find the class of Ω_{π_F} in $\mathrm{Pic}(\mathbb{F}l(k-1,k,k+1;n))$, which is an important ingredient in the computation of

$$\pi_{X,*}\ c_{\dim F_1(X)}(\mathcal{P}^d_\phi(\pi_X^* L)).$$

We also begin the computation of the above pushforward, translating it into a problem in Schubert calculus, at the end of this section.

Finally, in §4.4.5, we solve the resulting Schubert calculus problem, obtaining the following formula for the number of lines: setting

$$B_i := \begin{cases} \binom{n-1}{k}, & i = n-1, \\ \sum_{a=m_i}^{M_i} \binom{i+1}{a} \dfrac{(i+1-a)!}{(i-n-a)!}\, \dfrac{1}{(n-k-1-a)!\,(n+k-i+a)!}, & i > n-1, \end{cases}$$

where

$$m_i = \max(i-n-k,\, 0),$$

$$M_i = \min(i-n,\, n-k-1),$$

the number of lines that appear in a general linear system of degree d and projective dimension $d-(k+1)(n-k)-n+3$ is

$$\prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!} \sum_{i=n-1}^{2n-1} (-1)^{i-n+1} \sigma(d+1,\, d+1-i) \binom{d+1-i}{f-i} B_i\, d^{f-i} \frac{(f-i+1)!}{i-n+1},$$

where $\sigma(n,k)$ denote Stirling numbers of the first kind. The necessary hypotheses for this statement are set down as Theorem 4.12.

As an example, we compute that a degree 4 hypersurface in $\mathbb{G}(1,3)$ (that satisfies the hypotheses of Theorem 4.12) contains 1280 lines, verifying the count of §4.4.1.

4.1 Preliminaries

Fano variety of k-planes. Let $X \subset \mathbb{P}^N$ be a scheme with a fixed embedding into $\mathbb{P}^N$. The Fano variety of (projective) k-planes contained in X will be denoted by $F_k(X)$. The variety $F_k(X)$ may be defined (cf. [Huy, Ch. 3.1]) as the space representing the functor

$$(Sch/k)^{op} \to (Sets)$$

that maps a scheme B of finite type over k to the set of closed subschemes $Y \subset X \times B$ such that $\pi_2|_Y$ is flat, where $\pi_2 \colon X \times B \to B$ is the projection onto the second factor of $X \times B$, and such that every fibre Y_b, $b \in B$, is a projective k-plane contained in X.

We denote by $\mathcal{U}_k(X)$ the universal family over $F_k(X)$. (A proof that $F_k(X)$ and $\mathcal{U}_k(X)$ exist may be found in, for instance, [AK77, Proposition 3.1(i)].)

Thus, if B is any scheme of finite type over k, and $Y \subset X \times B$ is a closed subscheme such that $\pi_2|_Y$ is flat, and such that every fibre Y_b, $b \in B$, is a projective k-plane contained in X, then there is a unique map $B \to F_k(X)$ such that Y is equal to the fibre product

$$\begin{array}{ccc} Y = B \times_{F_k(X)} \mathcal{U}_k(X) & \longrightarrow & \mathcal{U}_k(X) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\exists!} & F_k(X) \end{array}$$

We follow traditional terminology in calling $F_k(X)$ a variety, but note that $F_k(X)$ may in fact not be reduced.

Grassmannians. A (projective) Grassmannian $\mathbb{G}(k,n)$ is a Fano variety of k-planes in the particular case when $X = \mathbb{P}^n$, embedded into itself by the identity map. We rapidly review the basic facts and notation around Grassmannians.

Let V be a finite-dimensional vector space over k. As usual, denote the Grassmannian of affine k-planes in V by $G(k,V)$ and the Grassmannian of projective k-planes in $\mathbb{P}V$ by $\mathbb{G}(k,\mathbb{P}V)$. (We have $G(k,V) = \mathbb{G}(k-1,\mathbb{P}V)$.) When $V = k^n$, we write $G(k,n)$ for $G(k,V)$ and $\mathbb{G}(k-1,n-1)$ for $\mathbb{G}(k-1,\mathbb{P}V)$.

We have identifications $G(1,V) \cong \mathbb{P}V$ and $G(\dim V - 1, V) \cong \mathbb{P}V^\vee$, where $V^\vee = \mathrm{Hom}_k(V,k)$ denotes the dual space of V. For other values of k, the Grassmannian $\mathbb{G}(k,n)$ is not isomorphic to a projective space.

More generally, given a point of $G(k,V)$ corresponding to an affine k-plane $\Lambda \subset V$, the short exact sequence $0 \to \Lambda \xrightarrow{i} V \to \mathrm{coker}(i) \to 0$ (where i is the inclusion map) dualizes to the short exact sequence $0 \to \ker(j) \to V^\vee \xrightarrow{j} \Lambda^\vee \to 0$, where $j = i^\vee$. The kernel $\ker(j)$ is a $(\dim V - k)$-plane in $V^\vee$, hence corresponds to a point of $G(\dim V - k, V^\vee)$ (it is equal to the annihilator $\Lambda^\perp \subset V^\vee$, that is, the subspace of equations vanishing on Λ). Similarly, given a $(\dim V - k)$-plane Λ' in $V^\vee$, we obtain a k-plane in V by taking the dual of the cokernel of the inclusion $\Lambda' \to V^\vee$ (the resulting k-plane may be thought of as the vanishing set of Λ'). We obtain a canonical identification $G(k,V) \cong G(\dim V - k, V^\vee)$ (and hence also $\mathbb{G}(k-1,\mathbb{P}V) \cong \mathbb{G}(\dim V - k - 1, \mathbb{P}V^\vee)$.

The variety $\mathbb{G}(k,n)$ is projective, smooth, irreducible, and has Picard rank 1 and dimension $(k+1)(n-k)$. The ample generator of the Picard group of $\mathbb{G}(k,n)$ is denoted by $\mathcal{O}_{\mathbb{G}}(1)$; $\mathcal{O}_{\mathbb{G}}(1)$ is in fact very ample, and the embedding associated to the complete linear system $|\mathcal{O}_{\mathbb{G}}(1)|$ is called the Plücker embedding, denoted by $Pl_{k,n}$. The Plücker embedding may also be described as the map $Pl_{k,n} : G(k+1,n+1) \to \mathbb{P}(\bigwedge^k V)$ that maps a given $k+1$-plane Λ corresponding to a point $[\Lambda]$ of $G(k+1,n+1)$ to $[w_1 \wedge \cdots \wedge w_{k+1}] \in \mathbb{P}(\bigwedge^k V)$, for a choice $\{w_1,\dots,w_{k+1}\}$ of a basis of Λ (this is well-defined since for another choice $\{w_1',\dots,w_{k+1}'\}$ of basis of Λ we have $w_1' \wedge \cdots \wedge w_{k+1}' = (\det \phi)\, w_1 \wedge \cdots \wedge w_{k+1}$, where $\phi \in \mathrm{Aut}\,\Lambda$ is the change-of-basis map). Thus, $Pl_{k,n}$ embeds $\mathbb{G}(k,n)$ into $\mathbb{P}^{\binom{n+1}{k+1}-1}$.

Over $G(k,n)$, there is a tautological rank k vector bundle S, together with an injective morphism $S \to \mathcal{O}^n_{G(k,n)}$ into a trivial bundle of rank n. Over the point $[\Lambda] \in G(k,n)$, the morphism $S_{[\Lambda]} \to (\mathcal{O}^n_{G(k,n)})_{[\Lambda]}$ corresponds to the inclusion of Λ into k^n. The tautological rank $n-k$ bundle Q is then defined to be the cokernel of the inclusion $S \to \mathcal{O}^n_{G(k,n)}$, so that we have an exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}^n_{G(k,n)} \longrightarrow Q \longrightarrow 0$$

as well as its dual sequence

$$0 \longrightarrow Q^\vee \longrightarrow \mathcal{O}^n_{G(k,n)} \longrightarrow S^\vee \longrightarrow 0.$$

By the Whitney formula, $c(S)c(Q) = c(\mathcal{O}^n_{G(k,n)}) = 1$, so $c(S) = s(Q)$ and $c(Q) = s(S)$ (and similar identities are easily derived for $S^\vee, Q^\vee$).

Let λ be a Young diagram fitting inside a box with $k+1$ rows and $(n-k)$ columns (equivalently, an integer partition $\lambda = (\lambda_0, \dots, \lambda_k)$ with $0 \le \lambda_k \le \lambda_{k-1} \le \dots \le \lambda_0 \le (n-k)$). Define a class σ_λ in the Chow ring $A^\bullet(\mathbb{G}(k,n))$ by the formula

$$\sigma_\lambda := \det\left(c_{\lambda_i+j-i}(Q)\right)_{0\le i,j\le k}; \tag{4.1}$$

the degree of σ_λ is the number $|\lambda|$ of boxes in λ (equivalently, is the sum $|\lambda| = \lambda_0 + \dots + \lambda_k$).

Let λ be a Young diagram fitting inside a $(k+1)\times(n-k)$ box, and let $a_i = n-k-\lambda_i+i$ for each $i = 0, \dots, k$. For a complete flag $(F_i)_{0\le i\le n}$ of linear subspaces of $\mathbb{P}^n$, the set

$$\Sigma_\lambda := \{[\Lambda] \in \mathbb{G}(k,n) : \dim(\Lambda \cap F_{a_i}) = i\} \subset \mathbb{G}(k,n)$$

is a locally closed subvariety such that $\left[\overline{\Sigma_\lambda}\right] = \sigma_\lambda$ (and does not depend on the particular choice of (F_i), up to rational equivalence).

Let a denote the Young diagram with a single row of a boxes (equivalently, the partition $(a,0,\dots,0)$), and let 1^b denote the Young diagram with a single column of b boxes (equivalently, the partition $(1,\dots,1,0,\dots,0)$, with b 1's). We have

$$\sigma_a = c_a(Q) \qquad \text{and} \qquad \sigma_{1^b} = c_b(S^\vee) = (-1)^b c_b(S),$$

where the first equality follows directly from the definition (4.1), and the second equality follows from the first by duality considerations (both equalities can also be interpreted in terms of the subvarieties Σ_λ).

The ring $A^\bullet(\mathbb{G}(k,n))$ is generated as an abelian group by σ_λ, where λ runs over all Young diagrams fitting inside a $(k+1)\times(n-k)$ box. The product of two classes $\sigma_\lambda, \sigma_\mu$ is then given by

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{|\nu|=|\lambda|+|\mu|} c^\nu_{\lambda,\mu}\sigma_\nu,$$

and $c^\nu_{\lambda,\mu}$ are in fact nonnegative integers known as the Littlewood–Richardson coefficients. There exist several algorithms for computing Littlewood–Richardson coefficients (one is [Ful98, Lemma 14.5.3]), but we will require only the following special cases:

- *Pieri's formulas.* The result of multiplying a class σ_λ by σ_a is a sum (with all coefficients equal to 1) over all classes corresponding to diagrams obtained by adding exactly a boxes to λ, with at most one box per column. Similarly, the result of multiplying a class σ_λ by σ_{1^b} is a sum (with all coefficients equal to 1) over all classes corresponding to diagrams obtained by adding exactly b boxes to λ, with at most one box per row.

  More precisely,

  Theorem (Pieri's formulas, [EH16], Proposition 4.9 and Theorem 4.14)**.** *We have*

$$\sigma_a \cdot \sigma_\lambda = \sum_{\substack{|\mu|=|\lambda|+a \\ \lambda_i \le \mu_i \le \lambda_{i-1}\,\forall i}} \sigma_\mu \qquad \textit{and} \qquad \sigma_{1^b}\cdot\sigma_\lambda = \sum_{\substack{|\mu|=|\lambda|+b \\ \lambda_i \le \mu_i \le \lambda_i+1 \;\; \forall i}} \sigma_\mu.$$

- *Classes of complementary dimension, [EH16, Proposition 4.6].* If $\sigma_\lambda, \sigma_\mu$ are classes in $A^\bullet(\mathbb{G}(k,n))$ with $|\lambda| + |\mu| = (k+1)(n-k)$, then

  $$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} [\text{pt.}], & \lambda \text{ is complementary to } \mu, \\ 0, & \text{otherwise,} \end{cases}$$

  where $\lambda = (\lambda_0, \ldots, \lambda_k)$ is said to be complementary to $\mu = (\mu_0, \ldots, \mu_k)$ if $\lambda_i + \mu_{k-i} = n-k$ for each $i = 0, \ldots, k$.

Much more information about $\mathbb{G}(k,n)$ can be found in, among many other sources, [EH16, Ch. 3–4] and [Ful98, Ch. 14].

Projective bundles. Let X be a scheme, E be a vector bundle of rank $r+1$ over X, and $\mathcal{E}$ be a sheaf of sections associated with E. Throughout this chapter, we denote the bundle of 1-dimensional subspaces

$$\pi : \mathbb{P}_{\text{sub}}(E) = \operatorname{Proj}(\operatorname{Sym} \mathcal{E}^\vee) \to X$$

in the fibres of E simply by $\mathbb{P}(E)$ (or $\mathbb{P}E$).

Over $\mathbb{P}E$, there is a tautological line bundle $\mathcal{O}_{\mathbb{P}E}(-1)$, equipped with an injective morphism to the vector bundle $\pi^* E$. In the fibre over $(x, [\ell])$, where $x \in X$ and ℓ is a line in the fibre E_x of E over x, this morphism is the inclusion

$$\mathcal{O}_{\mathbb{P}E}(-1)_{(x,[\ell])} \to (\pi^* E)_{(x,[\ell])} \qquad \cong \qquad \ell \to E_x.$$

The dual bundle of $\mathcal{O}_{\mathbb{P}E}(-1)$ is denoted by $\mathcal{O}_{\mathbb{P}E}(1)$.

The Picard group of $\mathbb{P}E$ is $\pi^* \operatorname{Pic}(X) \oplus \mathbb{Z} \cdot c_1(\mathcal{O}_{\mathbb{P}E}(1))$.

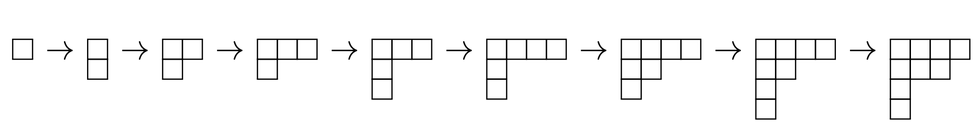
For much more, we refer to [EH16, Ch. 9] and [Ful98, App. B].

Hooklength formula. Define a partial order on Young diagrams by setting $\lambda \leq \mu$ if and only if λ is contained in μ (the corresponding partial order on integer partitions is: $\lambda = (\lambda_0, \lambda_1 \ldots,) \leq \mu = (\mu_0, \mu_1, \ldots)$ if and only if $\lambda_i \leq \mu_i$ for each $i \geq 0$).

The set of all Young diagrams equipped with the above partial order is a lattice, called the Young lattice. In the Hasse (directed) graph corresponding to the Young lattice, there is a directed edge from λ to μ if and only λ is obtained by adding a single box to μ (in general, the Hasse graph of a lattice contains a directed edge $x \to y$ whenever $x < y$ and there is no z such that $x < z < y$). A maximal chain from λ to μ in the Young lattice corresponds to the pair $(\{e_1, \ldots, e_k\}, \{\nu_1, \ldots, \nu_{k+1}\})$, where the e_i are directed edges and the ν_i are vertices in the Hasse graph, such that e_i is an edge from ν_i to ν_{i+1}, $\nu_1 = \lambda$, and $\nu_{k+1} = \mu$. From now on, we do not distinguish between maximal chains between pairs of elements in the Young lattice and the corresponding directed paths in the Hasse graph.

A (standard) Young tableau of shape λ is a filling of λ with elements of $A = \{1, \ldots, |\lambda|\}$ with no repetition such that the numbers are increasing along each row and along each column (the set A may be replaced by any other totally ordered set with $|\lambda|$ elements). For a fixed Young diagram λ, there is a bijection between Young tableaux of shape λ and directed paths from $\square$ to λ in the Young lattice; roughly speaking, the integer placed inside box b of the tableau records the order in which box b was adjoined to the diagram along the path from $\square$ to λ. For example, below, the tableau on the left corresponds to the path in the Young lattice on the right:

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 & 9 & \\ \cline{1-3} 5 & & & \\ \cline{1-1} 8 & & & \\ \cline{1-1} \end{array} \quad \leftrightarrow$$



In the above language, as a consequence of Pieri's formulas, we have

$$\sigma_1 \cdot \sigma_\lambda = \sum_{\substack{\exists \text{ edge } \lambda \to \mu \\ \text{in the Young lattice}}} \sigma_\mu$$

and consequently, by induction on k,

$$\sigma_1^k \cdot \sigma_\lambda = \sum_{\substack{\exists \text{ (directed) path } \lambda \to \mu \text{ of} \\ \text{length } k \text{ in the Young lattice}}} \sigma_\mu \qquad \text{for any } k \geq 1.$$

In particular, for any $k \geq 1$, we have the formula

$$\sigma_1^k = \sum_{|\mu| = k} f_\mu \, \sigma_\mu,$$

where for a Young diagram λ, f_λ denotes the number of (standard) Young tableaux of shape λ (equivalently, the number of directed paths from $\square$ to λ in the Young lattice).

There is a well-known expression for f_λ, which we now describe.

Let λ be a Young diagram (drawn according to the so-called English convention, with the number of boxes weakly decreasing along rows from the top to the bottom of the diagram), and let b be a box in λ. The hook of b in λ is the collection of boxes in λ that are either: in the same row and to the right of b, or in the same column and below b (together with b itself). The hooklength h_b of b in λ is the number of boxes contained in the hook of b.

For example, in the filling of λ below (on the right), the number in box b is equal to the hooklength of b in λ:

	6				

10	9	7	6	5	2
9	8	6	5	4	1
7	6	4	3	2	
6	5	3	2	1	
2	1				

Theorem (Hooklength formula)**.** *For any Young diagram λ, we have* $f_\lambda = \dfrac{|\lambda|!}{\prod_b h_b}$.

The hooklength formula is due originally to Frame, Robinson, and Thrall (whose proof was based on earlier independent works of Young, and Frobenius). There exist several proofs of the hooklength formula; the paper [GNW79] contains one of the nicest ones (by means of a random process defined on the Young diagram), as well as references to earlier work.

4.2 A ϕ-principal-parts-bundle method for counting lines in a linear system

Let $X \subset \mathbb{P}^N$ be a scheme with a fixed embedding into $\mathbb{P}^N$.

For the method described in this section, we will suppose that X satisfies the following two hypotheses:

(i) X is covered by lines, in the sense that for each point $x \in X$ there is a projective line ℓ with $x \in \ell \subset X$.

(ii) The Fano variety $F_1(X)$ of lines in X is irreducible. (Alternatively, it is also enough to assume that $F_1(X)$ is smooth, since then the method can proceed connected component by connected component.)

Degree of a line bundle with respect to a curve class. For a nonsingular curve $C \subset X$, set $\deg_C L := \deg(L|_C)$. Let $[\ell]$ be a point of $F_1(X)$, so that $\ell \subset X$ is a (projective) line contained in X. We obtain a continuous map

$$\begin{aligned} F_1(X) &\to \mathbb{Z} \\ [\ell] &\mapsto \deg_\ell L. \end{aligned}$$

As $F_1(X)$ is connected (in fact irreducible by hypothesis), this map is constant. At times below, we call the integer $\deg_\ell L$ simply the degree of L.

4.2.1 Definition of the set of lines that appear in a linear system

Let L be a line bundle on X, and let $H := |L| = \mathbb{P}(H^0(X, L))$ be the (projective) complete linear system associated to L. Let $\mathcal{H}$ be the universal family over H, and let $\pi : \mathcal{H} \to H$ be the projection map. Consider now an arbitrary linear system $|V| = \mathbb{P}(V) \subset H$ corresponding to a linear subspace $V \subset H^0(X, L)$. Let $i\colon |V| \to H$ be the inclusion, and set $\mathcal{V}$ to be the fibre product

$$\begin{tikzcd} \mathcal{V} := |V| \times_H \mathcal{H} \arrow[r] \arrow[d, "\pi_V"'] & \mathcal{H} \arrow[d, "\pi"] \\ |V| \arrow[r, "i"] & H \end{tikzcd}$$

Then the set

$$\left\{[\ell] \in F_1(Y) : Y = \pi_V^{-1}(y),\ y \in |V|\right\} = \bigcup_{y \in |V|} F_1(\pi_V^{-1}(y))$$

will be called the set of lines that appear in the linear system $|V|$.

In the present chapter, we are especially interested in situations when this set is finite, and in such a situation the goal is to count the number of points contained in this set.

4.2.2 Description of the method

Denote by $\pi_F : \mathcal{U}_1(X) \to F_1(X)$ the projection from the universal family of lines in X to the Fano variety of lines in X, and by $\pi_X : \mathcal{U}_1(X) \to X$ the projection to X:

$$\begin{tikzcd} \mathcal{U}_1(X) \arrow[r, "\pi_F", "\mathbb{P}^1\text{-bundle}"'] \arrow[d, "\pi_X"] & F_1(X) \\ X & \end{tikzcd} \tag{4.2}$$

Clearly, $\mathcal{U}_1(X)$ is a $\mathbb{P}^1$-bundle over $F_1(X)$, and the projection π_F is the corresponding projection.

The derivation ϕ. For the map π_F of (4.2), we have the usual exact sequence for the relative cotangent sheaf over $\mathcal{U}_1(X)$:

$$\pi_F^*(\Omega_{F_1(X)}) \longrightarrow \Omega_{\mathcal{U}_1(X)} \longrightarrow \Omega_{\pi_F} \longrightarrow 0.$$

The map π_F has one-dimensional fibres, and the sheaf Ω_{π_F} is in fact a line bundle. We take the derivation ϕ to be the map $\Omega_{\mathcal{U}_1(X)} \to \Omega_{\pi_F}$.

Dimension count. Let d denote the degree of the restriction of L to a line contained in X. Set $M := h^0(X, L)$, so that $|L| \cong \mathbb{P}^M$.

Let

$$\Gamma := \{([\ell], [H]) : H \in |L|,\ \ell \subset H\} \quad \subset \quad F_1(X) \times |L| \quad \cong \quad F_1(X) \times \mathbb{P}^M$$

(this is a projective algebraic variety). Let π_1 and π_2 be the projection maps from $F_1(X) \times \mathbb{P}^M$ to $F_1(X)$ and $\mathbb{P}^M$, respectively. We have the incidence correspondence

$$\begin{tikzcd} & \Gamma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ F_1(X) & & \mathbb{P}^M \end{tikzcd}$$

(where π_j, $j = 1, 2$, denote the restrictions of the two projections to Γ). (Note that we have $\pi_2^{-1}([H]) = F_1(H)$.)

Consider the fibre $\pi_1^{-1}([\ell])$. By restricting the equation of a hypersurface to ℓ (which gives a polynomial of degree d on $\ell \cong \mathbb{P}^1$), we obtain a restriction homomorphism

$$j_\ell : H^0(X, L) \to H^0(\ell, L|_\ell) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)).$$

Suppose that the following conditions are true:

(*) The projection π_1 is dominant.

(**) For each $[\ell] \in F_1(X)$, the map j_ℓ is surjective.

Because Γ is proper, (*) implies that π_1 is surjective. Instead of (*), it is also enough to suppose that the image of π_1 is irreducible, or that the image of π_1 is smooth.

By (**), we have the short exact sequence $0 \longrightarrow \ker j_\ell \longrightarrow k^{M+1} \xrightarrow{j_\ell} k^{d+1} \longrightarrow 0$, and $\pi_1^{-1}([\ell]) \cong \mathbb{P}(\ker j_\ell) \cong \mathbb{P}^{M-d-1}$. By (*), we conclude that π_1 is a map to an irreducible variety with irreducible fibres, and so Γ is also irreducible of dimension $\dim(F_1(X)) + M - d - 1$.

Suppose in addition that the following condition holds:

(***) The projection π_2 is generically finite-to-one onto its image.

Then (***) $\dim(\pi_2(\Gamma)) = \dim \Gamma$, so that $\dim(F_1(X)) - 1 \leq d$. The codimension of $\pi_2(\Gamma)$ in $\mathbb{P}^M$ is $d - \dim(F_1(X)) + 1$, and therefore a general linear system $|V| \subset \mathbb{P}^M$ of dimension $d - \dim(F_1(X)) + 1$ intersects $\pi_2(\Gamma)$ in finitely many points.

Proposition 4.1. *Suppose that conditions (*), (**), (***) hold. For any* $d \geq \dim(F_1(X)) - 1$*, finitely many lines appear in a general linear system of degree* d *and projective dimension* $d - \dim(F_1(X)) + 1$.

Translation to a Chern class computation. Recall that $\pi_X : \mathcal{U}_1(X) \to X$ denotes the projection. Fix an integer $d \geq \dim(F_1(X)) - 1$, and let $|V| \subset |L|$ be a linear system of projective dimension $d - \dim(F_1(X)) + 1$. The pullback $\pi_X^* V \subset H^0(\mathcal{U}_1(X), \pi_X^* L)$ will be denoted V once again, which is justified by the fact that $\pi_X^* : H^0(X, L) \to H^0(\mathcal{U}_1(X), \pi_X^* L)$ is an injective map (the latter is the case because by hypothesis X is covered by lines).

We have the following morphism of locally free sheaves on $\mathcal{U}_1(X)$.

$$V \otimes_k \mathcal{O}_{\mathcal{U}_1(X)} \longrightarrow \mathcal{P}^d_\phi(\pi_X^* L). \tag{4.3}$$

Let ℓ be a line in X, and let $\{(x, [\ell]) : x \in \ell\} = \pi_F^{-1}([\ell])$ be the corresponding line in $\mathcal{U}_1(X)$, which we again denote by ℓ. Under the isomorphism $\ell \cong \mathbb{P}^1$, the restriction $\Omega_{\pi_F}\big|_\ell$ is identified with $\Omega_{\mathbb{P}^1}$. By functoriality of $\mathcal{P}^r_\phi$ (Theorem 2.24), we then have

$$(\mathcal{P}^d_\phi(\pi_X^* L))\big|_\ell \cong \mathcal{P}^d(L|_{\mathbb{P}^1}) \cong \mathcal{P}^d(\mathcal{O}(d)),$$

where $\mathcal{P}^d(L|_{\mathbb{P}^1})$ denotes the usual (full) principal parts bundle on $\mathbb{P}^1$, and moreover the diagram

$$
\begin{tikzcd}
 & V \otimes_k \mathcal{O}_{\mathcal{U}_1(X)} \arrow[r] \arrow[d, "\text{restr.}"] & \mathcal{P}^d_\phi(\pi_X^* L) \arrow[d, "\text{restr. isom.}"] \\
V|_{\mathbb{P}^1} \otimes_k \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\text{isom.}"] & V \otimes_k \mathcal{O}_{\mathbb{P}^1} \arrow[r] & \mathcal{P}^d(\mathcal{O}(d))
\end{tikzcd} \tag{4.4}
$$

commutes, where $V|_{\mathbb{P}^1}$ denotes the linear subspace of $H^0(\mathbb{P}^1, \mathcal{O}(d))$ spanned by $\{ s|_{\mathbb{P}^1} : s \in V\}$.

Lemma 4.2. *The following are equivalent:*

(a) The line ℓ appears in the linear system $|V|$.

(b) For every point $x \in \ell$, the morphism $(V \otimes_k \mathcal{O}_{\mathcal{U}_1(X)})_{(x,[\ell])} \to (\mathcal{P}^d_\phi(\pi_X^* L))_{(x,[\ell])}$ of fibres over $(x, [\ell])$ does not have full rank.

(c) There is a point $x \in \ell$ such that the morphism $(V \otimes_k \mathcal{O}_{\mathcal{U}_1(X)})_{(x,[\ell])} \to (\mathcal{P}^d_\phi(\pi_X^* L))_{(x,[\ell])}$ of fibres over $(x, [\ell])$ does not have full rank.

Proof. By the commutativity of (4.4), and the fact that the right vertical map is an isomorphism, the morphism $(V \otimes_k \mathcal{O}_{\mathcal{U}_1(X)})_{(x,[\ell])} \to (\mathcal{P}^d_\phi(\pi_X^* L))_{(x,[\ell])}$ does not have full rank if and only if the morphism $(V \otimes_k \mathcal{O}_{\mathbb{P}^1})_x \to \mathcal{P}^d(\mathcal{O}(d))_x$ does not have full rank.

$(a) \Rightarrow (b)$: By definition, ℓ appears in $|V|$ only if there is a divisor H in $|V|$ that contains ℓ. Choosing an equation $h \in V$ of H, every element of $\operatorname{span}_k(h) \subset V$ vanishes on ℓ, and so every element of $\operatorname{span}(h)$ restricts to the zero section on ℓ. We conclude that for every $x \in \ell$, the subspace $(\operatorname{span}(h) \otimes_k \mathcal{O}_{\mathbb{P}^1})_x \subset (V \otimes_k \mathcal{O}_{\mathbb{P}^1})_x$ is contained in the kernel of the map $(V \otimes_k \mathcal{O}_{\mathbb{P}^1})_x \to \mathcal{P}^d(\mathcal{O}(d))_x$, which cannot then have full rank.

$(b) \Rightarrow (c)$: Immediate.

$(c) \Rightarrow (a)$: Suppose $0 \neq h \in V$ is sent to zero by the morphism $(V \otimes_k \mathcal{O}_{\mathbb{P}^1})_x \to \mathcal{P}^d(\mathcal{O}(d))_x$. In an affine chart $x \in \operatorname{Spec} k[t] \cong U \subset \ell$, h is a degree d polynomial in t; the image in $\mathcal{P}^d(\mathcal{O}(d))_x \cong k^{d+1}$ is $(h(x), h'(x), \ldots, h^{(d)}(x)) = 0$. Since $h^{(i)}(x) = 0$ for $i = 0, \ldots, d$, h is identically zero on U, hence is identically zero on $\mathbb{P}^1 \cong \ell$ by continuity. $\square$

The locus of lines in $\mathcal{U}_1(X)$ that appear in the linear system $|V|$ is thus identified with the locus of points of $\mathcal{U}_1(X)$ where the morphism (4.3) does not have full rank. To find the class of the latter in $A^\bullet(\mathcal{U}_1(X))$, we apply the Thom–Porteous formula:

Theorem ([Ful98, Theorem 14.4])**.** *Let $\sigma : E \to F$ be a morphism of vector bundles of ranks e and f on a Cohen–Macaulay scheme X, and let $k \le \min(e, f)$ be an integer. If the degeneracy locus*

$$D_k(\sigma) \coloneqq \{x \in X : \operatorname{rank}(\sigma_x : E_x \to F_x) \le r\}$$

has the expected codimension $(e-r)(f-r)$, then, setting $c(F)/c(E) \eqqcolon c$,

$$[D_k(\sigma)] = \det \begin{pmatrix} c_{f-r} & c_{f-r+1} & \cdots & c_{(f-r)+(e-r)-1} \\ c_{f-r-1} & c_{f-r} & \cdots & c_{(f-r)+(e-r)-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{f-e+1} & c_{f-e+2} & \cdots & c_{f-r} \end{pmatrix} \in A^{(e-r)(f-r)}(X). \tag{4.5}$$

In particular, we will need the following special case, in which the determinantal expression simplifies considerably:

Corollary. *Let F be a vector bundle of rank f on a Cohen–Macaulay scheme X, and let $1 \le i \le f$ be an integer. If the locus D where the vector bundle map*

$$\sigma : \mathcal{O}_X^{f-i+1} \to F$$

does not have full rank has the expected codimension i, then $[D] = c_i(F) \in A^i(X)$.

The Corollary follows from the Thom–Porteous formula: First, σ does not have full rank at x if and only if $\operatorname{rank}\big(\sigma_x : (\mathcal{O}_X^{f-i+1})_x \to F_x\big) \le f-i$, so that the expected codimension is equal to $(f-i+1-(f-i))(f-(f-i)) = i$. Next, $c(\mathcal{O}_X^{f-i+1}) = 1$, so (in the notation of the statement of Thom–Porteous) $c = c(F)$. Finally, since (again referring to the statement of Thom–Porteous for the notation) $(e-r) = 1$, the matrix in (4.5) is the 1×1 matrix $c_{f-r} = c_i$.

Applying the above corollary to morphism (4.3), with $f = d + 1$, $i = \dim(F_1(X)$, we obtain:

Theorem 4.3. *Suppose that* $\mathcal{U}_1(X)$ *is Cohen–Macaulay and* $F_1(X)$ *is either irreducible or smooth and connected, let* $d \geq \dim(F_1(X)) - 1$ *be an integer, and let* $|V| \subset |L|$ *be a linear system of degree* d *and projective dimension* $d - \dim(F_1(X)) + 1$.

If the locus of points of $\mathcal{U}_1(X)$ *where the morphism* $V \otimes_k \mathcal{O}_X \to \mathcal{P}^d_\phi(\pi^* L)$ *is not full rank has dimension one (equivalently, codimension* $(\dim F_1(X))$*), then the class in* $A_1(X)$ *of the locus of lines that appear in* $|V|$ *is given by*

$$(\pi_X)_* \, c_{\dim F_1(X)}(\mathcal{P}^d_\phi(\pi_X^* L)).$$

Remarks. (i) Since $\mathcal{U}_1(X)$ is a $\mathbb{P}^1$-bundle over $F_1(X)$, $\dim \mathcal{U}_1(X) = \dim(F_1(X)) + 1$, so having codimension $\dim(F_1(X))$ in $\mathcal{U}_1(X)$ is equivalent to having dimension one.

(ii) If (*), (**), (***) are true, and if $|V|$ is general (and of degree and dimension as in the statement of the theorem), then Proposition 4.1 shows that finitely many lines appear in $|V|$. In this case, the degeneracy locus of morphism (4.3) is indeed one-dimensional, and we can apply Theorem 4.3 to count the number of lines.

For $X = \mathbb{P}^n$ and $X = \mathbb{G}(k,n)$ (the two particular cases considered below), (*) and (**) are not difficult to verify, but (***) is not known in general.

We end this section with a minor comment regarding the initial hypotheses on X. Namely, we claim that if X contains a line, then we can assume without loss of generality that X is covered by lines.

Since $F_1(X)$ is proper, and π_F is a proper morphism, $\mathcal{U}_1(X)$ is proper, and so

$$\pi_X(\mathcal{U}_1(X)) =: Y \subset X$$

is a closed subscheme. For any H in $|V|$, a line $\ell \subset Y \subset X$ is contained in the restriction $H|_Y$ if and only if ℓ is contained in H.

If $\dim Y = 1$, then X contains a single line (namely Y), and it is simple to find the hypersurfaces in $|V|$ that contain Y (for example, by restricting the equations of a basis of V to Y and checking if they vanish). Suppose that $\dim Y \geq 2$. If Y is contained in some H in $|V|$, then H contains infinitely many lines. Because we will only be interested in $|V|$ in which only finitely many lines appear, we can restrict our attention to $|V|$ such that Y is not contained in H for any H in $|V|$. Restricting to Y, we have reduced to the case when X is covered by lines.

4.3 Projective spaces

Proceeding according to the method outlined in the previous section, we find an expression for the number of lines that appear in a linear system in projective space $\mathbb{P}^n$.

4.3.1 Fano variety of lines in $\mathbb{P}^n$ and its universal family

The Fano variety $F_1(\mathbb{P}^n)$ of lines in $\mathbb{P}^n$ is the (projective) Grassmannian $\mathbb{G}(1,n)$ (by definition of the two objects); the dimension of $F_1(\mathbb{P}^n)$ is $2n-2$.

The universal family over $\mathbb{G}(1,n)$ is isomorphic to the projective bundle $\mathbb{P}T_{\mathbb{P}^n}$. To see this (at least on the level of sets), note that for $x \in \mathbb{P}^n$, we have $(\mathbb{P}T_{\mathbb{P}^n})_x \cong \mathbb{P}(T_{\mathbb{P}^n,x}) \cong \mathbb{P}^n$, and a point $v_x \in (\mathbb{P}T_{\mathbb{P}^n})_x$ corresponds to the class of a line in $\mathbb{P}^n$ that passes through x and has the tangent direction v_x.

Both $\mathbb{G}(1,n)$ and $\mathbb{P}T_{\mathbb{P}^n}$ are smooth and irreducible.

Specializing the method of section 4.2 to the case of $\mathbb{P}^n$, we then have the maps

$$\begin{tikzcd} \mathbb{P}T_{\mathbb{P}^n} \arrow[r, "\pi_F", "\mathbb{P}^1\text{-bundle}"'] \arrow[d, "\pi"'] & \mathbb{G}(1,n) \\ \mathbb{P}^n & \end{tikzcd}$$

The bundle Ω_{π_F} can be identified with $\mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(1)$ (indeed, over a point $(x,[\ell])$ of $\mathbb{P}T_{\mathbb{P}^n}$, the fibre of Ω_{π_F} is $\Omega_{x,\ell}$, while the fibre of $\mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(-1)$ is $\ell \subset T_{x,\mathbb{P}^n}$, which can be identified with $T_{x,\ell}$, so Ω_{π_F} and $\mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(1)$ agree fibre by fibre, and this identification is algebraic).

The derivation is $\phi\colon \Omega_{\mathbb{P}T_{\mathbb{P}^n}} \to \Omega_{\pi_F} \cong \mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(1)$, and the class of lines contained in a linear system of degree d and projective dimension $d-2n+3$ is (expected to be) given by the degree of

$$\pi_* c_{2n-2}(\mathcal{P}^d_\phi(\pi^*\mathcal{O}_{\mathbb{P}^n}(d))).$$

4.3.2 Number of lines that appear in a linear system of hypersurfaces of degree d and projective dimension $d-2n+3$ in $\mathbb{P}^n$

Chern and Segre classes of $T_{\mathbb{P}^n}$. Let H denote the linear equivalence class of a hyperplane in $\mathbb{P}^n$. It is well-known that the Chow ring of $\mathbb{P}^n$ is

$$A^\bullet(\mathbb{P}^n) = \mathbb{Z}[H]/(H^{n+1}).$$

From the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

it follows by the Whitney formula that

$$c(T_{\mathbb{P}^n}) = (1+H)^{n+1}.$$

The total Segre class $s(T_{\mathbb{P}^n}) = 1 + s_1(T_{\mathbb{P}^n}) + s_2(T_{\mathbb{P}^n}) + \cdots$ may be found by formally inverting the total Chern class, so that

$$s(T_{\mathbb{P}^n}) = \frac{1}{c(T_{\mathbb{P}^n})} = \frac{1}{(1+H)^{n+1}} = \sum_{j=0}^{\infty}(-1)^j\binom{n+j}{n}H^j. \tag{4.6}$$

The computation. The general expression for $c_{2n-2}(\mathcal{P}^d_\phi(L))$ given in Theorem 2.29 specializes to

$$c_{2n-2}\left(\mathcal{P}^d_\phi\left(\pi^*\mathcal{O}_{\mathbb{P}^n}(d)\right)\right) = \sum_{k=0}^{2n-2}\sigma(d+1,\, d+1-k)\binom{d+1-k}{2n-2-k} c_1\left(\pi^*\left(\mathcal{O}_{\mathbb{P}^n}(d)\right)\right)^{2n-2-k} c_1(\mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(1))^k.$$

By the push-pull formula, for any $k = 0, \dots, 2n-2$,

$$\pi_*\left(\pi^*\left(c_1(\mathcal{O}_{\mathbb{P}^n}(d))^{2n-2-k}\right) \cdot c_1(\mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(1))^k\right) = c_1\left(\mathcal{O}_{\mathbb{P}^n}(d)\right)^{2n-2-k} \cdot \pi_*\left(c_1(\mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(1))^k\right).$$

By the definition of Segre classes, for a vector bundle $p\colon E \to X$ of rank e,

$$s_j(E) := p_*\left(c_1\left(\mathcal{O}_{\mathbb{P}E}(1)\right)^{e-1+j}\right).$$

Therefore, by (4.6) (together with the fact that $s_j(E) = 0$ for $j < 0$),

$$\pi_*\left(c_1(\mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}}(1))^k\right) = s_{k-n+1}(T_{\mathbb{P}^n}) = \begin{cases} (-1)^{k-n+1}\binom{k+1}{n}H^{k-n+1}, & k \geq n-1, \\ 0, & k < n+1. \end{cases}$$

Since

$$c_1\left(\mathcal{O}_{\mathbb{P}^n}(d)\right)^{2n-2-k} = (dH)^{2n-2-k},$$

we find that

$$\pi_*\left(c_{2n-2}\left(\mathcal{P}^d_\phi\left(\pi^*\mathcal{O}_{\mathbb{P}^n}(d)\right)\right)\right) = \left(\sum_{k=n-1}^{2n-2} \sigma(d+1,\, d+1-k)\binom{d+1-k}{2n-2-k} d^{2n-2-k}\,(-1)^{k-n+1}\binom{k+1}{n}\right)H^{n-1}.$$

This is (the rational equivalence class of) the locus of lines that appear in one of the members of the linear series, so it is a class in $A^{n-1}(\mathbb{P}^n)$. The number of lines in the locus may be found by intersecting $\pi_*\left(c_{2n-2}\left(\mathcal{P}^d_\phi\left(\pi^*\mathcal{O}_{\mathbb{P}^n}(d)\right)\right)\right)$ with another copy of H. so that we obtain the following theorem.

Theorem 4.4. *Let $n \geq 1$ and $d \geq 2n-3$ be integers, and $|V| \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ be a linear system of projective dimension $d - 2n + 3$. If the locus of points of $\mathbb{P}(T_{\mathbb{P}^n})$ where the morphism $V \otimes_k \mathcal{O}_{\mathbb{P}T_{\mathbb{P}^n}} \to \mathcal{P}^d_\phi(\pi_X^*\mathcal{O}(d))$ does not have full rank is one-dimensional, then the number of* *lines that appear in $|V|$ is*

$$\sum_{k=n-1}^{2n-2} (-1)^{k-n+1}\,\sigma(d+1,\,d+1-k)\binom{d+1-k}{2n-2-k}\binom{k+1}{n}d^{2n-2-k}. \tag{4.7}$$

Examples 1. *Cubic surface in* $\mathbb{P}^3$ $(d=3, n=3)$. Expression (4.7) in the statement of Theorem 4.4 is equal to

$$\sigma(4,2)\binom{3}{3}3^2 - \sigma(4,1)\binom{4}{3}3 + \sigma(4,0)\binom{5}{3} = 11\cdot 9 - 6\cdot 4\cdot 3 + 0 = 99 - 72 = 27.$$

Pencil of quartic surfaces in $\mathbb{P}^3$ $(d=4, n=3)$.

$$\sigma(5,3)\binom{3}{2}\binom{3}{3}4^2 - \sigma(5,2)\binom{2}{1}\binom{4}{3}4 + \sigma(5,1)\binom{1}{0}\binom{5}{3} = 320.$$

Net of quintic surfaces in $\mathbb{P}^3$ $(d=5,\ n=3)$.

$$\sigma(6,4)\binom{4}{2}\binom{3}{3}5^2 - \sigma(6,3)\binom{3}{1}\binom{4}{3}5 + \sigma(6,2)\binom{2}{0}\binom{5}{3} = 1990.$$

Quintic threefold in $\mathbb{P}^4$ $(d=5, n=4)$.

$$\sigma(6,3)\binom{4}{4}5^3 - \sigma(6,2)\binom{5}{4}5^2 + \sigma(6,1)\binom{6}{4}5 - \sigma(6,0)\binom{7}{4} = 2875.$$

Remark. The sum in Theorem 4.4 may also be written

$$\sum_{k=n-1}^{2n-2} (-1)^{k-n+1}\,J(d,k)\binom{d+1-k}{2n-2-k}\binom{k+1}{n}d^{2n-2-k}$$

using the functions $J(n,k)$ introduced in §2.6.

4.4 Grassmannians

Let V a vector space of dimension $n+1$ over k. The Grassmannian $\mathbb{G}(0,\mathbb{P}V)$ equals $\mathbb{P}V$, so that $F_1(\mathbb{G}(0,\mathbb{P}V)) = \mathbb{G}(1,\mathbb{P}V) \cong \mathbb{G}(1,n)$. Similarly, as $\mathbb{G}(n-1,\mathbb{P}V) = \mathbb{G}(0,\mathbb{P}V^\vee)$, we have

$F_1(\mathbb{G}(n-1,\mathbb{P}V)) = \mathbb{G}(1,\mathbb{P}V^\vee) \cong \mathbb{G}(1,n)$.

On the other hand, for $1 < k < n-1$, $\mathbb{G}(k,\mathbb{P}V)$ is not isomorphic to a projective space. For this section, we suppose that we are in the range $1 < k < n-1$. The approach taken in this section can be slightly modified to recover the results of the previous section for cases $k = 1$ and $k = n-1$.

For the remainder of this section, unless noted otherwise, we identify the Grassmannian $\mathbb{G}(k,n)$ with its image under the Plücker embedding.

We find an expression for the number of lines that appear in a linear system in a Grassmannian $\mathbb{G}(k,n)$, proceeding according to the method outlined in section 4.2.

4.4.1 Degree of the Fano variety of lines contained in the intersection of a quadric and a quartic in $\mathbb{P}^5$

The result of this section will be useful for checking a later computation; the section is not logically necessary for later sections.

The Grassmannian $\mathbb{G}(1,3) = G(2,4)$ is the first example of a Grassmannian that is not isomorphic to $\mathbb{P}^n$ for some n. The Plücker embedding $Pl_{1,3} : \mathbb{G}(1,3) \to \mathbb{P}\left(\bigwedge^2 \mathbb{C}^4\right) \cong \mathbb{P}^5$ sends the affine 2-plane spanned by a pair of vectors $\{v,w\}$ to the class of $v \wedge w$. A form $\omega \in \wedge^2 k^4$ is decomposable exactly when $\omega \wedge \omega = 0$, so that $\mathbb{G}(1,3)$ is cut out of $\mathbb{P}^5$ (with homogeneous coordinates $[x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34}]$) by the quadric equation $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$.

Since $Pl_{1,3}^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{G}(1,3)}(1)$, the image of a degree d hypersurface in the Grassmannian gets sent to the intersection of a degree d hypersurface in $\mathbb{P}^5$ with the image of $\mathbb{G}(1,3)$ under $Pl_{1,3}$. Therefore, the Fano variety of lines in $\mathbb{G}(1,3)$ may be realized as the intersection of the Fano variety of lines contained in a quadric hypersurface in $\mathbb{P}^5$ with the Fano variety of lines contained in a quartic hypersurface in $\mathbb{P}^5$.

Let X and Y be quadric and quartic hypersurfaces, respectively, in $\mathbb{P}^5$. We follow the approach of [EH16], §6.2 (pp. 198–201) for computing the classes of the Fano varieties of lines of X and Y in the Chow ring of the Grassmannian of lines in $\mathbb{P}^5$. The key proposition is

Proposition (Proposition 6.4 of [EH16], p. 198)**.** *Let V be a finite-dimensional vector space, and let S be the taultological (rank $(k+1)$) subbundle on the Grassmannian $\mathbb{G}(k,\mathbb{P}V)$ of projective k-planes in $\mathbb{P}V$. Let g be a form of degree d on $\mathbb{P}V$, and let $X = V(g)$ be the hypersurface cut out by g. Then g gives rise to a global section s_g of $\operatorname{Sym}^d S^\vee$, whose zero locus is $F_k(X)$, the Fano variety of k-planes in X. Consequently, when $F_k(X)$ has the expected codimension $\binom{k+d}{k} = \operatorname{rank}(\operatorname{Sym}^d S^\vee)$ in $\mathbb{G}(k,\mathbb{P}V)$, we have*

$$[F_k(X)] = c_{\binom{k+d}{k}}(\operatorname{Sym}^d S^\vee) \in A^\bullet(\mathbb{G}(k,\mathbb{P}V)).$$

The fibre of the tautological subbundle S over the point parametrizing the k-plane Λ in $\mathbb{G}(k,\mathbb{P}V)$ is the k-plane Λ itself and, roughly speaking, the section s_g of Proposition 4.4.1 is obtained by restricting the global form g to Λ as the base point varies in $\mathbb{G}(k,\mathbb{P}V)$.

The Chow ring of $\mathbb{G}(1,5)$ is additively generated by σ_λ indexed by Young diagrams that fit inside the box $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline\end{array}$.

The total Chern class of the dual bundle of the tautological subbundle $S^\vee$ on $\mathbb{G}(1,5)$ is $c(S^\vee) = 1 + \sigma_{\square} + \sigma_{\substack{\square\\\square}}$. Applying the splitting principle, we suppose that $S^\vee$ splits as the direct sum of two line bundles $L \oplus M$. Writing $c_1(L) := \lambda$ and $c_1(M) := \mu$, we then have

$$c(S^\vee) = (1+\lambda)(1+\mu) = 1 + (\lambda+\mu) + \lambda\mu,$$

so that

$$\lambda + \mu = \sigma_{\square} \qquad \text{and} \qquad \lambda \cdot \mu = \sigma_{\substack{\square\\\square}}.$$

Continuing to assume that $S^\vee$ splits as $L \oplus M$, the bundles $\operatorname{Sym}^2(S^\vee)$ and $\operatorname{Sym}^4(S^\vee)$ then split as

$$\begin{aligned}
&\mathrm{Sym}^2(S^\vee) = L^2 \oplus (L \otimes M) \oplus M^2 \qquad \text{and} \\
&\mathrm{Sym}^4(S^\vee) = L^4 \oplus (L^3 \otimes M) \oplus (L^2 \otimes M^2) \oplus (L \otimes M^3) \oplus M^4.
\end{aligned}$$

The total Chern classes are then

$$\begin{aligned}
&c(\mathrm{Sym}^2(S^\vee)) = (1+2\lambda)(1+\lambda+\mu)(1+2\mu) \qquad \text{and} \\
&c(\mathrm{Sym}^4(S^\vee)) = (1+4\lambda)(1+3\lambda+\mu)(1+2\lambda+2\mu)(1+\lambda+3\mu)(1+4\mu).
\end{aligned}$$

The Chern classes appearing in Proposition 4.4.1 above are

$$\begin{aligned}
c_3(\mathrm{Sym}^2(S^\vee)) &= 2\lambda(\lambda+\mu)\,2\mu = 4(\lambda+\mu)\lambda\mu = 4\sigma_{\square}\sigma_{\genfrac{}{}{0pt}{}{\square}{\square}} \qquad \text{and} \\
c_5(\mathrm{Sym}^4(S^\vee)) &= 4\lambda(3\lambda+\mu)(2\lambda+2\mu)(\lambda+3\mu)4\mu \\
&= 32(\lambda+\mu)\lambda\mu(3\lambda+\mu)(\lambda+3\mu) \\
&= 32(\lambda+\mu)\lambda\mu(3(\lambda+\mu)^2+4(\lambda\mu)) \\
&= 32\sigma_{\square}\sigma_{\genfrac{}{}{0pt}{}{\square}{\square}}(3\sigma_{\square}^2+4\sigma_{\genfrac{}{}{0pt}{}{\square}{\square}}).
\end{aligned}$$

Pieri's formulas (§4.1) are applied several times in the following computations. We have

$$\sigma_{\square}\cdot\sigma_{\genfrac{}{}{0pt}{}{\square}{\square}} = \sigma_{\genfrac{}{}{0pt}{}{\square\square}{\square}}$$

(as $\sigma_{\genfrac{}{}{0pt}{}{\square}{\genfrac{}{}{0pt}{}{\square}{\square}}} = 0$ in $A^\bullet(\mathbb{G}(1,5))$), so that

$$c_3(\mathrm{Sym}^2(S^\vee)) = 4\sigma_{\genfrac{}{}{0pt}{}{\square\square}{\square}}.$$

Next, we compute the class

$$c_5(\mathrm{Sym}^4(S^\vee)) = 32\sigma_{\ydiagram{1}}\sigma_{\ydiagram{1,1}}(3\sigma_{\ydiagram{1}}^2 + 4\sigma_{\ydiagram{1,1}}).$$

To begin, we have

$$\sigma_{\ydiagram{1}}^2 = \sigma_{\ydiagram{1,1}} + \sigma_{\ydiagram{2}}, \qquad \text{so that} \qquad 3\sigma_{\ydiagram{1}}^2 + 4\sigma_{\ydiagram{1,1}} = 7\sigma_{\ydiagram{1,1}} + 3\sigma_{\ydiagram{2}}.$$

Then, since

$$\sigma_{\ydiagram{1}} \cdot \sigma_{\ydiagram{1,1}} = \sigma_{\ydiagram{2,1}} \qquad \text{and} \qquad \sigma_{\ydiagram{1}} \cdot \sigma_{\ydiagram{2}} = \sigma_{\ydiagram{3}} + \sigma_{\ydiagram{2,1}},$$

we have

$$\sigma_{\ydiagram{1}} \cdot (7\sigma_{\ydiagram{1,1}} + 3\sigma_{\ydiagram{2}}) = 7\sigma_{\ydiagram{2,1}} + 3\sigma_{\ydiagram{3}} + 3\sigma_{\ydiagram{2,1}} = 10\sigma_{\ydiagram{2,1}} + 3\sigma_{\ydiagram{3}}.$$

To compute $c_5(\mathrm{Sym}^4(S^\vee))$, we need the facts that

$$\sigma_{\ydiagram{1,1}} \cdot \sigma_{\ydiagram{2,1}} = \sigma_{\ydiagram{3,2}} \qquad \text{and} \qquad \sigma_{\ydiagram{1,1}} \cdot \sigma_{\ydiagram{3}} = \sigma_{\ydiagram{4,1}},$$

so

$$c_5(\mathrm{Sym}^4(S^\vee)) = 32\sigma_{\ydiagram{1,1}}(10\sigma_{\ydiagram{2,1}} + 3\sigma_{\ydiagram{3}}) = 320\sigma_{\ydiagram{3,2}} + 30\sigma_{\ydiagram{4,1}}.$$

The class of the Fano variety of lines contained in the intersection of X and Y (which will be a multiple of the point class) may be found by taking the product $[F_1(X)] \cdot [F_1(Y)]$. Since (by the laws for intersecting Schubert classes of complementary dimension)

$$\sigma_{\ydiagram{2,1}} \cdot \sigma_{\ydiagram{3,2}} = \sigma_{\ydiagram{4,4}} \qquad \text{and} \qquad \sigma_{\ydiagram{2,1}} \cdot \sigma_{\ydiagram{4,1}} = 0,$$

we have

$$c_3(\mathrm{Sym}^2(S^\vee)) \cdot c_5(\mathrm{Sym}^4(S^\vee)) = 4\sigma_{\ydiagram{2,1}}(320\sigma_{\ydiagram{3,2}} + 30\sigma_{\ydiagram{4,1}}) = 1280\sigma_{\ydiagram{4,4}} = 1280\,[\mathrm{pt.}]\,,$$

where [pt.] denotes the class of a point in the Chow ring.

4.4.2 Picard group of flag varieties of type $\mathbb{F}l(k, k+1, \ldots, k+m; n)$

Fix an integer $n \geq 1$, and a vector space V of dimension $n+1$ over k. For the remainder of this section, unless mentioned otherwise, Λ_k denotes a projective plane in $\mathbb{P}V$ of dimension k, and Λ'_k the corresponding affine plane in V of dimension $k+1$.

A (projective, $m+1$-step) flag variety $\mathbb{F}l(k_0, k_1, \ldots, k_m; n)$ $(0 \leq m < n,\ 0 \leq k_0 < \cdots < k_m < n)$ is the variety parametrizing $m+1$-tuples of projective planes $([\Lambda_{k_0}], [\Lambda_{k_1}], \ldots [\Lambda_{k_m}])$ such that $\Lambda_{k_0} \subset \Lambda_{k_1} \subset \cdots \subset \Lambda_{k_m}$.

The reason for our interest in flag varieties here is that (as will follow from the results of this section) the universal family over the Fano variety of lines in a Grassmannian $\mathbb{G}(k, n)$ can be identified with the flag variety $\mathbb{F}l(k-1, k, k+1; n)$. For some of the computations in the sequel, it will be useful to describe more generally the Picard group of a flag variety of type $\mathbb{F}l(k, k+1, \ldots, k+m; n)$ (with $0 \leq k < n,\ 0 \leq m < n-k$).

First, it is clear that $\mathbb{F}l(k; n) = \mathbb{G}(k, n)$, so that $\mathrm{Pic}(\mathbb{F}l(k; n)) \cong \mathbb{Z}$.

Next, suppose that $1 \leq m$. For $0 \leq i \leq m$, consider the projection

$$\hat{p}_i : \mathbb{F}l(k, \ldots, k+m; n) \to \mathbb{F}l(k, \ldots, \widehat{k+i}, \ldots, k+m; n).$$

The fibre of $\hat{p}_i$ over the point $([\Lambda_k], \ldots, [\Lambda_{k+i-1}], [\Lambda_{k+i+1}], \ldots [\Lambda_{k+m}])$ is

$$([\Lambda_k], \ldots, [\Lambda_{k+i-1}], [\Lambda_{k+i}], [\Lambda_{k+i+1}], \ldots [\Lambda_{k+m}]) \in \ \mathbb{F}l(k, \ldots, k+m; n).$$

In particular, in the fibre, $[\Lambda_{k+i}]$ is the class of a projective $(k+i)$-plane such that

$$\begin{cases} \Lambda_k \subset \Lambda_{k+1}, & j = 0, \\ \Lambda_{k+i-1} \subset \Lambda_{k+i} \subset \Lambda_{k+i+1}, & 0 < i < m, \\ \Lambda_{k+m-1} \subset \Lambda_{k+m}, & j = m. \end{cases}$$

If $i = 0$, the fibre consists of all hyperplanes in Λ_{k+1}, so can be identified with $\mathbb{P}^{k+1}$; if $i = m$, the fibre can be identified with lines in k^{n+1}/Λ'_{k+m-1}, and hence in turn identified with $\mathbb{P}^{n-k-m}$; if $0 < i < m$, the fibre can be identified with lines in $\Lambda'_{k+i+1}/\Lambda'_{k+i-1}$, hence with $\mathbb{P}^1$. So $\hat{p}_i$ is a fibration with projective space fibres, and it follows that

$$\operatorname{Pic}(\mathbb{F}l(k,\ldots,k+m;n)) \cong \operatorname{Pic}(\mathbb{F}l(k,\ldots,\widehat{k+i},\ldots,k+m;n)) \oplus \mathbb{Z}.$$

By induction on m,

$$\operatorname{Pic}(\mathbb{F}l(k,k+1,\ldots,k+m;n)) \cong \mathbb{Z}^{m+1}.$$

Remark. Arbitrary flag varieties can be similarly seen to be compositions of fibrations with Grassmannian fibres over a Grassmannian, so that the Picard group of any flag variety with $m+1$ steps is isomorphic to $\mathbb{Z}^{m+1}$.

Basis of $\operatorname{Pic}(\mathbb{F}l(k,k+1,\ldots,k+m;n))$**.** It is convenient to set $\Lambda_{-1} = \varnothing$ and $\Lambda'_{-1} = 0$. Fix $0 \le k < n,\ 0 \le m < n-k$, and consider the flag variety $\mathbb{F}_{k,m} := \mathbb{F}l(k,k+1,\ldots,k+m;n)$.

For $0 \le i \le m$, introduce a curve $\ell_{k+i} \subset \mathbb{F}_{k,m}$ as follows: fix projective planes

$$\Lambda^*_{k-1} \subset \Lambda^*_k \subset \cdots \subset \Lambda^*_{k+i-1} \subset \Lambda^*_{k+i+1} \subset \cdots \subset \Lambda^*_{k+m} \subset \Lambda^*_{k+m+1}$$

and set

$$\ell_{k+j} := \left\{([\Lambda_k],\ldots[\Lambda_{k+m}]) \in\ \mathbb{F}l(k,\ldots,k+m;n) : \Lambda_{k+j} = \Lambda^*_{k+j}\ \ \forall j \neq i\right\}.$$

The points of ℓ_{k+i} correspond to $(k+i)$-planes Λ_{k+i} with $\Lambda^*_{k+i-1} \subset \Lambda_{k+i} \subset \Lambda^*_{k+i+1}$, which are parametrized by lines in $\Lambda'^*_{k+i+1}/\Lambda'^*_{k+i-1}$, hence by $\mathbb{P}^1$, and it follows that ℓ_{k+i} is a rational curve in $\mathbb{F}_{k,m}$.

For $0 \le j \le m$, let $\pi_{k+j} : \mathbb{F}_{k,m} \to \mathbb{G}(k+j,n)$ denote the projection. For $i \neq j$, we see that $\pi_{k+j}(\ell_{k+i}) = \left[\Lambda^*_{k+j}\right]$ is a point; on the other hand, for $i = j$, $\pi_{k+i}(\ell_{k+i})$ is the curve $\{[\Lambda_{k+i}] : \Lambda^*_{k+i-1} \subset \Lambda_{k+i} \subset \Lambda^*_{k+i+1}\} \subset \mathbb{G}(k+i,n)$, which we denote by $\tilde{\ell}_{k+i}$. The restriction of

π_{k+i} to ℓ_{k+i} has degree one.

Lemma 4.5. *We have* $\mathcal{O}_{\mathbb{G}(k+i,n)}(1)\big|_{\tilde{\ell}_{k+i}} = \mathcal{O}_{\tilde{\ell}_{k+i}}(1)$. *Therefore, under the Plücker embedding of* $\mathbb{G}(k+i,n)$, $\tilde{\ell}_{k+i}$ *is sent to a projective line.*

Proof. By functoriality of pullback, $\mathcal{O}_{\mathbb{G}}(1)|_{\tilde{\ell}_{k+i}} = (Pl_{k+i,n}|_{\tilde{\ell}_{k+i}})^* \mathcal{O}_{\mathbb{P}}(1)$, so that the equality $\mathcal{O}_{\mathbb{G}(k+i,n)}(1)\big|_{\tilde{\ell}_{k+i}} = \mathcal{O}_{\tilde{\ell}_{k+i}}(1)$ is equivalent the image of $\tilde{\ell}_{k+i}$ under the Plücker embedding being a curve of degree one, hence a projective line.

It is enough to check the first claim for a particular curve $\tilde{\ell}_{k+i}$, since any choice of $\Lambda'^*_{k-1} \subset \Lambda'^*_{k+1}$ can be continuously moved to another by actions of elements of $\mathrm{GL}(V)$.

Let $\{v_0, \dots, v_n\}$ be a basis of V, and let $\Lambda'^*_{k+i-1} = \mathrm{span}(v_0, \dots, v_{k+i-1})$ and $\Lambda'^*_{k+i+1} = \mathrm{span}(v_0, \dots, v_{k+i+1})$. As observed above, affine $k+i+1$-planes Λ'_{k+i} with $\Lambda'^*_{k+i-1} \subset \Lambda'_{k+i} \subset \Lambda'^*_{k+i+1}$ correspond to lines in $\Lambda'^*_{k+i+1}/\Lambda'^*_{k+i-1} \cong \mathrm{span}(v_{k+i}, v_{k+i+1})$, hence are in bijection with $\mathbb{P}(\mathrm{span}(v_{k+i}, v_{k+i+1})) \cong \mathbb{P}^1$. Explicitly, the correspondence is

$$[\alpha : \beta] \in \mathbb{P}^1 \leftrightarrow [\mathrm{span}(v_0, v_1, \dots, v_{k+i-1}, \alpha v_{k+i} + \beta v_{k+i+1})] \in \tilde{\ell}_{k+i} \subset \mathbb{G}(k+i,n).$$

In turn, a plane of the form $\mathrm{span}(v_0, v_1, \dots, v_{k+i-1}, \alpha v_{k+i} + \beta v_{k+i+1})$ is sent to

$$[v_0 \wedge \cdots \wedge v_{k+i-1} \wedge (\alpha v_{k+i} + \beta v_{k+i+1})] = [\alpha(v_0 \wedge \cdots \wedge v_{k+i-1} \wedge v_{k+i}) + \beta(v_0 \wedge \cdots \wedge v_{k+i-1} \wedge v_{k+i+1})] \in \mathbb{P}(\bigwedge^{k+i+1} V),$$

so that the image of $\tilde{\ell}_{k+i}$ in $\mathbb{P}(\wedge^{k+i+1}V)$ is a projective line, and by the remark at the start of the proof it follows that $\mathcal{O}_{\mathbb{G}(k+i,n)}(1)\big|_{\tilde{\ell}_{k+i}} = \mathcal{O}_{\tilde{\ell}_{k+i}}(1)$. □

For each $0 \le i \le m$, set $L_{k+i} = \pi^*_{k+i}\mathcal{O}_{\mathbb{G}(k+i,n)}(1)$. We use the previously introduced notation $\deg_\ell L := \deg L|_\ell$ for a nonsingular curve $\ell \subset X$ and line bundle L over a scheme X.

Lemma 4.6. *For all* $0 \le i, j \le m$, *we have*

$$\deg_{\ell_{k+j}} L_{k+i} = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

Proof. Fix $0 \le i \le m$.

Suppose first that $j = i$. Let D_{k+i} be a divisor in $\left|\mathcal{O}_{\mathbb{G}(k+i,n)}(1)\right|$. Under the Plücker embedding, D_{k+i} is sent to a hyperplane section of $\mathbb{G}(k+i,n)$ while $\tilde{\ell}_{k+i}$ is sent to a projective line (and therefore their intersection is a point). The degree of $L_{k+i}|_{\ell_{k+i}}$ is $(\pi^*_{k+i}D_{k+i}) \cdot \ell_{k+i}$. By the push-pull formula, we have

$$\pi_{k+i,*}((\pi^*_{k+i}D_{k+i}) \cdot \ell_{k+i}) = D_{k+i} \cdot \pi_{k+i,*}\ell_{k+i} = D_{k+i} \cdot \tilde{\ell}_{k+i} = 1.$$

On the other hand, the restriction of π_{k+i} to ℓ_{k+i} is a degree one morphism onto its image, so it follows that $(\pi^*_{k+i}D_{k+i}) \cdot \ell_{k+i} = 1$, hence that $\deg_{\ell_{k+i}} L_{k+i} = 1$.

Suppose that $j \neq i$. Let D_{k+i} be a divisor in $\left|\mathcal{O}_{\mathbb{G}(k+i,n)}(1)\right|$ that does not pass through $\pi_{k+i}(\ell_{k+j}) = [\Lambda_{k+i}] \in \mathbb{G}(k+i,n)$ (such a divisor clearly exists, since there are plenty of hyperplanes that do not pass through a point). By functoriality of pullback,

$$L_{k+i}|_{\ell_{k+j}} = (\pi_{k+i}|_{\ell_{k+j}})^* \mathcal{O}_{\mathbb{G}(k+i,n)}(1).$$

Then

$$\deg_{\ell_{k+j}} L_{k+i} = ((\pi_{k+i}|_{\ell_{k+j}})^* D_{k+i}) \cdot \ell_{k+j} = (\pi_{k+i}|_{\ell_{k+j}})^* (D_{k+i} \cdot [\Lambda_{k+i}]) = 0.$$

□

Corollary 4.7. *The map* $d\colon \mathbb{Z}^{m+1} \to \operatorname{Pic}(\mathbb{F}_{k,m})$ *that sends* $(d_0, \dots, d_m) \in \mathbb{Z}^{m+1}$ *to the line bundle* $d_0L_k + \cdots + d_mL_{k+m}$ *is a group isomorphism.*

Proof. Composing d with the map $\delta\colon \operatorname{Pic}(\mathbb{F}_{k,m}) \to \mathbb{Z}^{m+1}$ given by

$$\delta\colon L \mapsto (\deg_{\ell_k} L, \deg_{\ell_{k+1}} L, \dots, \deg_{\ell_{k+m}} L),$$

we obtain the identity map on $\mathbb{Z}^{m+1}$ by Lemma 4.6, hence d is injective. Moreover, δ splits the short exact sequence $0 \to \mathbb{Z}^{m+1} \to \operatorname{Pic}(\mathbb{F}_{k,m}) \to \operatorname{coker}(d) \to 0$, and since $\operatorname{Pic}(\mathbb{F}_{k,m}) \cong \mathbb{Z}^{m+1}$, it is not hard to see that then $\operatorname{coker}(d) = 0$, so d is also surjective. □

We have shown that the set $\{\pi^*_{k+i}\mathcal{O}_{\mathbb{G}(k+i,n)}(1)\}_{i=0}^m$ is a basis of $\operatorname{Pic}(\mathbb{F}_{k,m})$. We refer to this basis as the Grassmannian basis from now on, unless otherwise stated.

4.4.3 Fano variety of lines in $\mathbb{G}(k,n)$ and its universal family

By Lemma 4.5, a pair

$$(\Lambda_{k-1}, \Lambda_{k+1}) \in \mathbb{G}(k-1,n) \times \mathbb{G}(k+1,n)$$

with $\Lambda_{k-1} \subset \Lambda_{k+1}$ determines a projective line in $\mathbb{G}(k,n)$.

The converse is true as well. Indeed, suppose that[1] $\ell \subset G(k,n)$ is a projective line under the Plücker embedding, and let $\{[\Lambda], [\Pi]\}$ be a basis of the k^2 corresponding to $\ell \cong \mathbb{P}^1$. We claim that $\Lambda \cap \Pi$ is an affine $k-1$-plane. To see this, set $\dim(\Lambda \cap \Pi) = d$. Let $\{x_1, \dots, x_d\}$ be a basis of $\Lambda \cap \Pi$, and complete it to bases $\{x_1, \dots, x_d, y_{d+1}, \dots, y_k\}$ and $\{x_1, \dots, x_d, z_{d+1}, \dots, z_k\}$ of Λ and Π, respectively. If $d < k-1$, then for $\alpha \cdot \beta \neq 0$, the element

$$\alpha(x_1 \wedge \cdots \wedge y_{k-1} \wedge y_k) + \beta(x_1 \wedge \cdots \wedge z_{k-1} \wedge z_k)$$

is not decomposable, so the corresponding point of ℓ is not contained in the image of the Plücker embedding. So $d = k-1$. It is clear that $\Lambda \cap \Pi$ is a subspace of every element of ℓ. It is also clear that $\dim(\Lambda + \Pi) = k+1$, and that every plane in ℓ is contained in $\Lambda + \Pi$. The converse is proven.

It follows that the Fano variety $F_1(\mathbb{G}(k,n))$ of lines in $\mathbb{G}(k,n)$ is isomorphic (at least as sets, but in fact as schemes as well) to the flag variety $\mathbb{F}l(k-1,k+1;n)$. The universal family $\mathcal{U}_1(\mathbb{G}(k,n))$ is isomorphic to the flag variety $\mathbb{F}l(k-1,k,k+1;n)$ (which is smooth, being a homogeneous space).

Proposition 4.8. *For any n and $0 \leq k \leq n-1$, the Fano variety $F_1(\mathbb{G}(k,n))$ of lines in $\mathbb{G}(k,n)$ is irreducible of dimension $(k+1)(n-k)+n-2$.*

[1]The Grassmannian of affine planes is used for the argument, since this slightly simplifies the notation.

Proof. For $k = 0$, $F_1(\mathbb{G}(0,n)) = \mathbb{G}(1,n)$, so is irreducible of dimension $2(n-1) = 1(n-0)+n-2$. Similarly, for $k = n-1$, $F_1(\mathbb{G}(n-1,n))$ is again $\mathbb{G}(1,n)$, so is irreducible of dimension $2(n-1) = ((n-1)+1)(n-(n-1))+n-2$.

For $0 < k < n-1$, $F_1(\mathbb{G}(k,n)) = \mathbb{F}l(k-1,k+1;n)$. The fibre of $\mathbb{F}l(k-1,k+1;n) \to \mathbb{F}l(k-1;n) = \mathbb{G}(k-1,n)$ over $[\Lambda_{k-1}]$ can be identified with the set of affine 2-planes in k^{n+1}/Λ'_{k-1}, so can be identified with $\mathbb{G}(1,n-k)$. So $\mathbb{F}l(k-1,k+1;n) \to \mathbb{G}(k-1,n)$ is a $\mathbb{G}(1,n-k)$-fibration, which shows that $\mathbb{F}l(k-1,k+1;n)$ is irreducible (as both the base and the fibre are irreducible) of dimension

$$\begin{aligned}\dim \mathbb{G}(k-1,n) + \dim \mathbb{G}(1,n-k) &= k(n-k+1) + 2(n-k-1) \\ &= (k+1)(n-k) + n - 2.\end{aligned}$$

□

The set-up of section 4.2 specializes to

$$\begin{array}{ccc} \mathbb{F}l(k-1,k,k+1;n) & \xrightarrow[\mathbb{P}^1\text{-bundle}]{\pi_F} & \mathbb{F}l(k-1,k+1;n) = F_1(\mathbb{G}(k,n)) \\ \Big\downarrow{\scriptstyle \pi} & & \\ \mathbb{G}(k,n), & & \end{array}$$

the derivation is $\phi\colon \Omega_{\mathbb{F}l(k-1,k,k+1;n)} \to \Omega_{\pi_F}$, and the Chow class of the locus of lines that appear in a linear system of degree d and projective dimension $d-(k+1)(n-k)-n+3$ is (expected to be) $\pi_* c_{(k+1)(n-k)+n-2}(\mathcal{P}^d_\phi(\pi^*\mathcal{O}_{\mathbb{G}}(d)))$.

4.4.4 The class of Ω_{π_F} in $\mathrm{Pic}(\mathbb{F}l(k-1,k,k+1;n))$.

We have the following commutative diagram.

(4.8)

$$\begin{tikzcd}
& & \mathbb{F}l(k-1,k,k+1;n) \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\mathbb{P}^1"] \arrow[dd, "\pi_k"] \arrow[ddll, bend right, "\pi_{k-1}"'] \arrow[ddrr, bend left, "\pi_{k+1}"] & & \\
& \mathbb{F}l(k-1,k;n) \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\mathbb{P}^1"'] & & \mathbb{F}l(k,k+1;n) \arrow[dl, "\mathbb{P}^1"] \arrow[dr, "\mathbb{P}^1"] & \\
\mathbb{G}(k-1,n) & & \mathbb{G}(k,n) & & \mathbb{G}(k+1,n)
\end{tikzcd}$$

where $\mathbb{P}^1$ along an arrow indicates that the corresponding morphism is a $\mathbb{P}^1$-fibration.

By Corollary 4.7, $\mathrm{Pic}(\mathbb{F}l(k-1,k,k+1;n))$ is a free abelian group of rank 3 with basis

$$\{\pi_{k-1}^*\mathcal{O}_{\mathbb{G}(k-1,n)}(1),\ \pi_k^*\mathcal{O}_{\mathbb{G}(k,n)}(1),\ \pi_{k+1}^*\mathcal{O}_{\mathbb{G}(k+1,n)}(1)\},$$

which as before we refer to as the Grassmannian basis of Pic.

Lemma 4.9. *Let* $\pi_F : \mathbb{F}l(k-1,k,k+1;n) \to \mathbb{F}l(k-1,k+1;n)$ *be the projection. The following identity holds in* $\mathrm{Pic}(\mathbb{F}l(k-1,k,k+1;n)$*:*

$$\Omega_{\pi_F} = \pi_{k-1}^*\mathcal{O}_{\mathbb{G}(k-1,n)}(1) \otimes \pi_k^*\mathcal{O}_{\mathbb{G}(k,n)}(-2) \otimes \pi_{k+1}^*\mathcal{O}_{\mathbb{G}(k+1,n)}(1).$$

Proof. Continue with the notation of §4.4.2. In particular, L_i denotes the line bundle $\pi_i^*\mathcal{O}_{\mathbb{G}(i,n)}(1)$, ℓ_{k-1} denotes the curve

$$\{([\Lambda_{k-1}], [\Lambda_k^*], [\Lambda_{k+1}^*]) : \Lambda_{k-2}^* \subset \Lambda_{k-1} \subset \Lambda_k^* \subset \Lambda_{k+1}^*\} \subset \mathbb{F}l(k-1,k,k+1;n)$$

for a fixed choice of projective planes Λ_{k-2}^*, Λ_k^*, Λ_{k+1}^*, and curves ℓ_k and ℓ_{k+1} are constructed similarly. We have $\deg_{\ell_i} L_j = 1$ if $i = j$ and 0 if $i \neq j$.

Writing the class of Ω_{π_F} in $\mathrm{Pic}(\mathbb{F}l(k-1,k,k+1;n))$ in terms of the Grassmannian basis,

$$\Omega_{\pi_F} = a\,L_{k-1} + b\,L_k + c\,L_{k+1} \qquad \text{for some } a,b,c \in \mathbb{Z},$$

and the coefficients may be found as degrees of restrictions of Ω_{π_F} to the curves ℓ_i: namely, $a = \deg_{\ell_{k-1}} \Omega_{\pi_F}$, $b = \deg_{\ell_k} \Omega_{\pi_F}$, $c = \deg_{\ell_{k+1}} \Omega_{\pi_F}$.

$b = -2$. Indeed, $\ell_k = \pi_F^{-1}(([\Lambda_{k-1}^*],[\Lambda_{k+1}^*]))$ is a fibre of π_F, so $\Omega_{\pi_F}\big|_{\ell_k} = \Omega_{\ell_k} = \mathcal{O}_{\ell_k}(-2)$ (since the cotangent bundle of $\mathbb{P}^1$ has degree -2).

$a = 1$. Consider the diagram

$$\begin{tikzcd} \mathbb{F}l(k-1,k,k+1;n) \arrow[r, "\pi_{k,k+1}"] \arrow[d, "\pi_F", "\mathbb{P}^1"'] & \mathbb{F}l(k,k+1;n) \\ \mathbb{F}l(k-1,k+1;n) & \end{tikzcd}$$

Let Π_k^* and Π_{k+1}^* be fixed planes. The fibre of $\pi_{k,k+1}$ over $([\Pi_k^*],[\Pi_{k+1}^*])$ consists of $k-1$-planes Λ_{k-1} contained in Π_k^*, which are in correspondence with $\mathbb{P}(\Pi_k'^*)^\vee \cong \mathbb{P}^k$.

The curve ℓ_{k-1} corresponding to the pair $\Lambda_{k-2}^* \subset \Lambda_k^*$ is contained in fibres of the type $\pi_{k,k+1}^{-1}(([\Lambda_k^*],[\Pi_{k+1}^*]))$, where Π_{k+1}^* is an arbitrary $k+1$-plane containing Λ_k^*. Fix a choice of ℓ_{k-1} and a choice of fibre of $\pi_{k,k+1}^{-1}(([\Lambda_k^*],[\Lambda_{k+1}^*])) \cong \mathbb{P}^k$ containing ℓ_{k-1}.

We show that $T_{\pi_F}\big|_{\mathbb{P}^k} = \mathcal{O}_{\mathbb{P}^k}(-1)$, which will be enough to show that $a = 1$. We recall that the tangent space to a Grassmannian $G(k,V)$ at $[\Lambda]$ is canonically identified with $\mathrm{Hom}_k(\Lambda, V/\Lambda)$. At a point $x = ([\Lambda_{k-1}],[\Lambda_k^*],[\Lambda_{k+1}^*])$ (which is contained in the fibre of π_F over the point of $\mathbb{F}l(k-1,k+1;n)$ corresponding to $\Lambda_{k-1} \subset \Lambda_{k+1}^*$), the relative tangent space $(T_{\pi_F})_x$ along the fibre is then identified with

$$\mathrm{Hom}(\Lambda_k'^*/\Lambda_{k-1}',\, \Lambda_{k+1}'^*/\Lambda_k'^*) \cong (\Lambda_k'^*/\Lambda_{k-1}')^\vee.$$

(although the last isomorphism depends on a choice of an isomorphism $\Lambda_{k+1}'^*/\Lambda_k'^* \cong k$, this choice can be made consistently with varying Λ_{k-1}', since both $\Lambda_k'^*$ and $\Lambda_{k+1}'^*$ are fixed). Since $(\Lambda_k'^*/\Lambda_{k-1}')^\vee$ is the dual of the cokernel

$$0 \to \Lambda_{k-1}' \to \Lambda_k'^* \to \Lambda_k'^*/\Lambda_{k-1}' \to 0,$$

it corresponds to a choice of line in $(\Lambda_k'^*)^\vee$, namely the line of linear functionals on $\Lambda_k'^*$ that vanish on Λ_{k-1}'. But the latter is exactly the fibre of the line bundle $\mathcal{O}_{\mathbb{P}(\Lambda_k'^{*\vee})}(-1)$ over x. This then yields an algebraic fibrewise identification of $T_{\pi_F}\Big|_{\mathbb{P}^k}$ with $\mathcal{O}_{\mathbb{P}^k}(-1)$.

Dualizing, we find that $\Omega_{\pi_F}\Big|_{\mathbb{P}^k} \cong \mathcal{O}_{\mathbb{P}^k}(1)$, and so $\deg_{\ell_{k-1}} \Omega_{\pi_F} = \deg_{\ell_{k-1}} \mathcal{O}_{\mathbb{P}^k}(1) = 1$, which shows that $a = 1$.

$c = 1$. The computation is similar to the case $a = 1$. Consider the diagram

$$\begin{tikzcd} \mathbb{F}l(k-1,k,k+1;n) \arrow[r, "\pi_{k-1,k}"] \arrow[d, "\mathbb{P}^1"', "\pi_F"] & \mathbb{F}l(k-1,k;n) \\ \mathbb{F}l(k-1,k+1;n) & \end{tikzcd}$$

The points of a fibre of $\pi_{k-1,k}$ over $([\Lambda_{k-1}^*], [\Lambda_k^*]) \in \mathbb{F}l(k-1,k;n)$ correspond to lines in V/Λ_k^*, hence the fibre can be identified with $\mathbb{P}(V/\Lambda_k^*) \cong \mathbb{P}^{n-k-1}$.

On the other hand, the fibre of the relative tangent bundle T_{π_F} over

$$x = ([\Lambda_{k-1}^*], [\Lambda_k^*], [\Lambda_{k+1}])$$

is identified with

$$\mathrm{Hom}(\Lambda_k'^*/\Lambda_{k-1}'^*, \Lambda_{k+1}'/\Lambda_k'^*) \cong \mathrm{Hom}(k, \Lambda_{k+1}'/\Lambda_k'^*) = \Lambda_{k+1}'/\Lambda_k'^*$$

(the isomorphism $\Lambda_k'^*/\Lambda_{k-1}'^* \cong k$ can again clearly be chosen consistently as Λ_{k+1}' varies). The latter is exactly the fibre of $\mathcal{O}_{\mathbb{P}(V/\Lambda_k'^*)}(-1)$ over x, and after dualizing, we similarly conclude that $\Omega_{\pi_F}\Big|_{\mathbb{P}^{n-k-1}} \cong \mathcal{O}_{\mathbb{P}^{n-k-1}}(1)$, so $\deg_{\ell_{k+1}} \Omega_{\pi_F} = \deg_{\ell_{k+1}} \mathcal{O}_{\mathbb{P}^{n-k-1}}(1) = 1$, which shows that $c = 1$.

$\square$

Another basis of $\mathrm{Pic}(\mathcal{U}_1(\mathbb{G}(k,n)))$**.** Let S and Q be the tautological sub- and quotient-bundles over $\mathbb{G}(k,n)$.

Over $[\Lambda] \in \mathbb{G}(k,n)$, we have the exact sequence of fibres

$$0 \to S_{[\Lambda]} \to (\mathcal{O}_{\mathbb{G}}^{n+1})_{[\Lambda]} \to Q_{[\Lambda]} \to 0 \qquad \cong \qquad 0 \to \Lambda \to k^{n+1} \to k^{n+1}/\Lambda \to 0.$$

A point in $\mathbb{P}_{\text{sub}}(k^{n+1}/\Lambda)$ corresponds to a line in k^{n+1}/Λ, hence an affine $k+2$ plane Λ' containing Λ. Projectivizing, we obtain a projective $(k+1)$-plane with $\mathbb{P}\Lambda \subset \mathbb{P}\Lambda'$. Conversely, a projective $k+1$-plane containing Λ determines a point of $\mathbb{P}_{\text{sub}}(k^{n+1}/\Lambda)$, and we have the following identification (as projective bundles over $\mathbb{G}(k,n)$).

$$\mathbb{F}l(k,k+1;n) \cong \mathbb{P}_{\text{sub}}(Q).$$

Dually, a point of $\mathbb{P}(\Lambda^\vee)$ can be identified with an affine hyperplane in Λ, i.e. an affine plane Λ'' of codimension one in Λ. We obtain the following identification of projective bundles over $\mathbb{G}(k,n)$:

$$\mathbb{F}l(k-1,k;n) \cong \mathbb{P}_{\text{sub}}(S^\vee).$$

Finally, we also have the identification $\mathbb{F}l(k-1,k,k+1;n) \cong \mathbb{P}(S^\vee) \times_{\mathbb{G}(k,n)} \mathbb{P}(Q)$. After making these identifications, we have the commutative diagram

$$\begin{tikzcd}
 & & \mathbb{P}(S^\vee) \times_{\mathbb{G}(k,n)} \mathbb{P}(Q) \arrow[dl, "q_1"'] \arrow[dd, "\pi"] \arrow[dr, "q_2"] & & \\
 & \mathbb{P}(S^\vee) \arrow[dl, "r_1"'] \arrow[dr, "p_1"'] & & \mathbb{P}(Q) \arrow[dl, "p_2"] \arrow[dr, "r_2"] & \\
\mathbb{G}(k-1,n) & & \mathbb{G}(k,n) & & \mathbb{G}(k+1,n)
\end{tikzcd} \tag{4.9}$$

We would next like to demonstrate that the line bundles $q_1^*\mathcal{O}_{\mathbb{P}S^\vee}(1)$, $q_2^*\mathcal{O}_{\mathbb{P}Q}(1)$, together with $\pi^*\mathcal{O}_{\mathbb{G}(k,n)}(1)$, are another basis of $\operatorname{Pic}(\mathbb{F}l(k-1,k,k+1;n))$ (which will be more convenient for the compuation of the number of lines).

Lemma 4.10. *In* $\operatorname{Pic}(\mathbb{F}l(k-1,k,k+1;n)) \cong \operatorname{Pic}(\mathbb{P}S^\vee \times_{\mathbb{G}(k,n)} \mathbb{P}Q)$*, we have the following* *identities.*

$$q_1^*\mathcal{O}_{\mathbb{P}S^\vee}(1) = \pi_{k-1}^*\mathcal{O}_{\mathbb{G}(k-1,n)}(1) \otimes \pi_k^*\mathcal{O}_{\mathbb{G}(k,n)}(-1),$$
$$q_2^*\mathcal{O}_{\mathbb{P}Q}(1) = \pi_{k+1}^*\mathcal{O}_{\mathbb{G}(k+1,n)}(1) \otimes \pi_k^*\mathcal{O}_{\mathbb{G}(k,n)}(-1).$$

Proof. We prove the first identity; the proof of the second is similar. By commutativity of (4.9) and functoriality of pullback, it is enough to demonstrate the identity

$$\mathcal{O}_{\mathbb{P}S^\vee}(1) = r_1^*\mathcal{O}_{\mathbb{G}(k-1,n)}(1) \otimes p_1^*\mathcal{O}_{\mathbb{G}(k,n)}(-1) \qquad \text{in } \operatorname{Pic}(\mathbb{F}l(k-1,k;n) \cong \operatorname{Pic}(\mathbb{P}S^\vee).$$

By Corollary 4.7, $\operatorname{Pic}(\mathbb{F}l(k-1,k;n))$ is isomorphic to $\mathbb{Z}^2$ with basis

$$\{r_1^*\mathcal{O}_{\mathbb{G}(k-1,n)}(1),\ p_1^*\mathcal{O}_{\mathbb{G}(k,n)}(1)\},$$

so that we have

$$\mathcal{O}_{\mathbb{P}S^\vee}(1) = r_1^*\mathcal{O}_{\mathbb{G}(k-1,n)}(a) \otimes p_1^*\mathcal{O}_{\mathbb{G}(k,n)}(b) \qquad \text{for some } a, b \in \mathbb{Z}. \tag{4.10}$$

Let ℓ_r by the curve in $\mathbb{F}l(k-1,k;n)$ that was denoted ℓ_{k-1} in §4.4.2 (so that ℓ_r is determined by a choice of projective planes $\Lambda_{k-2}^* \subset \Lambda_k^*$, $r_1(\ell_r) =: \tilde{\ell}_r \cong \mathbb{P}^1$ and $p_1(\ell_r) = [\Lambda_k^*]$); similarly, let ℓ_p the curve in $\mathbb{F}l(k-1,k;n)$ that was denoted ℓ_k (so that ℓ_p is determined by a choice of projective planes $\Lambda_{k-1}^* \subset \Lambda_{k+1}^*$, $r_1(\ell_p) = [\Lambda_{k-1}^*]$ and $p_1(\ell_p) =: \tilde{\ell}_p \cong \mathbb{P}^1$).

By Lemma 4.6,

$$\deg r_1^*\mathcal{O}_{\mathbb{G}(k-1,n)}(1)\big|_{\ell_r} = 1, \quad \deg p_1^*\mathcal{O}_{\mathbb{G}(k,n)}(1)\big|_{\ell_r} = 0,$$
$$\deg r_1^*\mathcal{O}_{\mathbb{G}(k-1,n)}(1)\big|_{\ell_p} = 0, \quad \deg p_1^*\mathcal{O}_{\mathbb{G}(k,n)}(1)\big|_{\ell_p} = 1,$$

so we see that $a = \deg \mathcal{O}_{\mathbb{P}S^\vee}(1)|_{\ell_r}$ and $b = \deg \mathcal{O}_{\mathbb{P}S^\vee}(1)|_{\ell_p}$ (where a and b are as in the equality (4.10)).

We first show that $a = 1$. $\mathbb{P}S^\vee$ is a projective bundle over $\mathbb{G}(k,n)$, and ℓ_r is contained in the fibre of $\mathbb{P}S^\vee$ over $\left[\Lambda_k^*\right]$, namely $\mathbb{P}(\Lambda_k^{*\vee})$, which is naturally identified with $\mathbb{G}(k-1,\Lambda_k^*)$ (ℓ_r corresponds to the pencil of $k-1$-planes in $\mathbb{G}(k-1,\Lambda_k^*)$ that contain Λ_{k-2}^*). A general divisor in $|\mathcal{O}_{\mathbb{P}S^\vee}(1)|$ will intersect each fibre of $\mathbb{P}S^\vee \to \mathbb{G}(k,n)$ in a projective hyperplane, and it follows that the intersection of ℓ_r with a general divisor in $|\mathcal{O}_{\mathbb{P}S^\vee}(1)|$ is equal to a point. This shows that $a = \deg \mathcal{O}_{\mathbb{P}S^\vee}(1)|_{\ell_r} = 1$.

We now argue that $b = -1$. Recall that $\tilde{\ell}_p = p_1(\ell_p)$ denotes the image of ℓ_p in $\mathbb{G}(k,n)$ ($\tilde{\ell}_p$ is a line in $\mathbb{G}(k,n)$ determined by $\Lambda_{k-1}^* \subset \Lambda_{k+1}^*$). Let P_{k-1} denote a trivial vector bundle of rank k over $\tilde{\ell}_p$, together with an injective morphism $P_{k-1} \to S|_{\tilde{\ell}_p}$ that in the fibre over the point $[\Lambda_k] \in \mathbb{G}(k,n)$ corresponding to the projective k-plane Λ is given by

$$(P_{k-1})_{[\Lambda_k]} \to \left(S|_{\tilde{\ell}_p}\right)_{[\Lambda_k]} \qquad \cong \qquad \Lambda_{k-1}^* \subset \Lambda_k.$$

Now the cokernel of $P_{k-1} \to S|_{\tilde{\ell}_p}$ is a line bundle over $\tilde{\ell}_p = \mathbb{P}^1$, hence is $\mathcal{O}_{\tilde{\ell}_p}(d)$ for some integer d. By additivity of degree on short exact sequences of vector bundles, we know that $d = \deg(S|_{\tilde{\ell}_p})$ (since a direct sum of trivial bundles has degree 0). On the other hand, since $c_1(S) = -\sigma_1$ is the negative of the class of a hyperplane section, we have

$$\deg(S|_{\tilde{\ell}_p}) = c_1(S) \cdot \tilde{\ell}_p = -1.$$

Therefore, we have the following short exact sequence of vector bundles.

$$0 \longrightarrow \oplus^k \mathcal{O}_{\tilde{\ell}_p} \longrightarrow S|_{\tilde{\ell}_p} \longrightarrow \mathcal{O}_{\tilde{\ell}_p}(-1) \longrightarrow 0. \tag{4.11}$$

But we have

$$\begin{aligned}\operatorname{Ext}^1(\mathcal{O}_{\tilde{\ell}_p}(-1), \oplus^k \mathcal{O}_{\tilde{\ell}_p}) &= \operatorname{Ext}^1(\mathcal{O}_{\tilde{\ell}_p}, \oplus^k \mathcal{O}_{\tilde{\ell}_p}(1)) \\ &= \oplus^k H^1(\mathcal{O}_{\tilde{\ell}_p}(1)) = 0,\end{aligned}$$

so (4.11) splits, so that $S|_{\tilde{\ell}_p} \cong \left(\oplus^k \mathcal{O}_{\tilde{\ell}_p}\right) \oplus \mathcal{O}_{\tilde{\ell}_p}(-1)$. Taking duals, $S^\vee|_{\tilde{\ell}_p} \cong \left(\oplus^k \mathcal{O}_{\tilde{\ell}_p}\right) \oplus \mathcal{O}_{\tilde{\ell}_p}(1)$.

On the other hand, if E is a vector bundle over a scheme X, and L is a subbundle of E of rank one, then $\mathcal{O}_{\mathbb{P}E}(-1)|_{\mathbb{P}(L)} \cong L$. Therefore, since $\mathbb{P}(\mathcal{O}_{\tilde{\ell}_p}(1)) = \tilde{\ell}_p$, we have $\mathcal{O}_{\mathbb{P}S^\vee}(-1)|_{\tilde{\ell}_p} \cong \mathcal{O}_{\tilde{\ell}_p}(1)$, and the conclusion follows upon taking duals.

□

Therefore $\{q_1^*\mathcal{O}_{\mathbb{P}S^\vee}(1),\ \pi^*\mathcal{O}_{\mathbb{G}}(1),\ q_2^*\mathcal{O}_{\mathbb{P}Q}(1)\}$ is another basis for $\operatorname{Pic}(\mathbb{F}l(k-1,k,k+1;n))$. We have

$$\Omega_{\pi_F} = q_1^*\mathcal{O}_{\mathbb{P}S^\vee}(1) \otimes q_2^*\mathcal{O}_{\mathbb{P}Q}(1).$$

Set

$$H_1 = q_1^* c_1(\mathcal{O}_{\mathbb{P}S^\vee}(1)), \quad H_2 = \pi^* c_1(\mathcal{O}_{\mathbb{G}}(1)), \quad H_3 = q_2^* c_1(\mathcal{O}_{\mathbb{P}Q}(1)).$$

Lemma 4.11. *For any integers a, b, c such that $a + k \geq 0$, $b \geq 0$, $n - k - 1 + c \geq 0$,*

$$\pi_*(H_1^{k+a}\, H_2^b\, H_3^{n-k-1+c}) = \begin{cases} (-1)^{a+c}\,\sigma_a\,\sigma_1^b\,\sigma_{1^c}, & a \geq 0 \text{ and } c \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $c_1(\mathcal{O}_{\mathbb{G}}(1)) = \sigma_1$, by the push-pull formula we have

$$\pi_*(H_1^{k+a}\, H_2^b\, H_3^{n-k-1+c}) = \pi_*\left(H_1^{k+a}\, H_3^{n-k-1+c}\right)\sigma_1^b.$$

By functoriality of pushforward (and push-pull once again), the latter is equal to

$$p_{1,*}\left[q_{1,*}\left(H_1^{k+a}\, H_3^{n-k-1+c}\right)\right]\sigma_1^b = p_{1,*}\left[q_{1,*}\left(H_3^{n-k-1+c}\right)c_1(\mathcal{O}_{\mathbb{P}S^\vee}(1))^{k+a}\right]\sigma_1^b$$

Since p_1, p_2 are flat and projective, by [Ful98, Proposition 1.7], for any $\alpha \in A^\bullet(\mathbb{P}Q)$ we have the identity

$$q_{1,*}q_2^*\alpha = p_1^*p_{2,*}\alpha.$$

In particular,

$$q_{1,*}H_3^{n-k-1+c} = q_{1,*}q_2^*c_1(\mathcal{O}_{\mathbb{P}Q}(1))^{n-k-1+c} = p_1^*p_{2,*}c_1(\mathcal{O}_{\mathbb{P}Q}(1))^{n-k-1+c}.$$

Applying the push-pull formula once again, we have

$$p_{1,*}\left[q_{1,*}\left(H_3^{n-k-1+c}\right)c_1(\mathcal{O}_{\mathbb{P}S^\vee}(1))^{k+a}\right]\sigma_1^b = p_{1,*}\left(c_1(\mathcal{O}_{\mathbb{P}S^\vee}(1))^{k+a}\right)\ p_{2,*}\left(c_1(\mathcal{O}_{\mathbb{P}Q}(1))^{n-k-1+c}\right)\ \sigma_1^b.$$

Since $S^\vee$ is a rank $k+1$-bundle and Q is a rank $n-k$-bundle over $\mathbb{G}(k,n)$,

$$p_{1,*}c_1(\mathcal{O}_{\mathbb{P}S^\vee}(1)^{k+a}) = \begin{cases} s_a(S^\vee), & a \geq 0, \\ 0, & a < 0, \end{cases} \qquad \text{and} \qquad p_{2,*}c_1(\mathcal{O}_{\mathbb{P}Q}(1)^{n-k-1+c}) = \begin{cases} s_c(Q), & c \geq 0, \\ 0, & c < 0. \end{cases}$$

By the tautological exact sequence for S and Q it follows that $c(S)c(Q) = 1$, so $s(Q) = c(S)$ and $s_c(Q) = c_s(S) = (-1)^c\sigma_{1^c}$. Similarly, $s(S^\vee) = c(Q^\vee)$, so that $s_a(S^\vee) = c_a(Q^\vee) = (-1)^a\sigma_a$, and the statement of the lemma follows. □

4.4.5 Number of lines that appear in a linear system of hypersurfaces of Plücker degree d and projective dimension $d-(k+1)(n-k)-n+3$ in $\mathbb{G}(k,n)$

Set $f = \dim F_1(\mathbb{G}(k,n)) = (k+1)(n-k)+n-2$. The general formula for the fth chern class given in Theorem 2.29 specializes to

$$c_f\left(\mathcal{P}_\phi^d\left(\pi^*\mathcal{O}_{\mathbb{G}}(d)\right)\right) = \sum_{i=0}^{f}\sigma(d+1,\, d+1-i)\binom{d+1-i}{f-i}c_1\left(\pi^*\left(\mathcal{O}_{\mathbb{G}}(d)\right)\right)^{f-i}\ c_1(\mathcal{O}_{\Omega_{\pi_F}})^i$$

and

$$c_1\left(\pi^*\left(\mathcal{O}_{\mathbb{G}}(d)\right)\right)^{f-i} c_1(\mathcal{O}_{\Omega_{\pi_F}})^i = d^{f-i} H_2^{f-i} (H_1 + H_3)^i = d^{f-i} \sum_{j=0}^{i} \binom{i}{j} H_1^j H_2^{f-i} H_3^{i-j}$$

so that

$$\pi_* c_f\left(\mathcal{P}^d_\phi\left(\pi^*\mathcal{O}_{\mathbb{G}}(d)\right)\right) = \sum_{i=0}^{f} \sigma(d+1,\, d+1-i)\binom{d+1-i}{f-i} d^{f-i} \sum_{j=0}^{i} \binom{i}{j} \pi_*(H_1^j H_2^{f-i} H_3^{i-j}).$$

To compute the number of lines, we need to multiply $\pi_* c_f\left(\mathcal{P}^d_\phi\left(\pi^*\mathcal{O}_{\mathbb{G}}(d)\right)\right)$ (the class of the locus of lines) by another copy of the hyperplane class σ_1, and it will be beneficial to carry out this multiplication at the present stage.

We first focus on the terms

$$\begin{aligned}\sigma_1 \cdot \pi_*(H_1^j H_2^{f-i} H_3^{i-j}) &= \sigma_1 \cdot \pi_*(H_1^{k+(j-k)} H_2^{f-i} H_3^{n-k-1+(i-j-n+k+1)}) \\ &= (-1)^{i-n+1} \sigma_{j-k} \sigma_1^{f-i+1} \sigma_{1^{i-j-n+k+1}} \qquad \text{when } j \geq k \quad \text{and} \quad i-j \geq n-k-1.\end{aligned}$$

The last term is a class of degree $\dim \mathbb{G}(k,n)$ (as it should be). We turn our attention to the following subproblem: compute the triple product

$$\sigma_a \sigma_1^b \sigma_{1^c}, \qquad a+b+c = \dim \mathbb{G}(k,n).$$

By either of Pieri's formulas, the product $\sigma_a \sigma_{1^c}$ (with $a > 0$, $c > 0$) has just two terms:

$$\sigma_a \sigma_{1^c} = \sigma_{a+1,1^{c-1}} + \sigma_{a,1^c} \qquad \text{(for example,} \quad \sigma_{\ydiagram{4}} \cdot \sigma_{\ydiagram{1,1,1}} = \sigma_{\ydiagram{5,1,1}} + \sigma_{\ydiagram{4,1,1,1}}).$$

Both of the terms are hooks. Let $\lambda_{\alpha,\beta} = (\alpha+1, 1^\beta)$ denote the hook with α horizontal boxes and β vertical boxes, not counting the corner box (for example, $\lambda_{3,2} = \ydiagram{4,1,1}$), such that $0 \leq \alpha \leq n-k-1$ and $0 \leq \beta \leq k$. Note that $|\lambda_{\alpha,\beta}| = \alpha + \beta + 1$.

We have reduced to the problem: compute

$$\sigma_1^b \sigma_{\lambda_{\alpha,\beta}} \qquad b + \alpha + \beta + 1 = \dim \mathbb{G}(k, n).$$

On the other hand, as described in §4.1, we have $\sigma_1^b = \sum_{|\mu|=b} f_\mu \, \sigma_\mu$, where f_μ is the number of standard Young tableaux with shape μ. By the law for multiplying Schubert classes of complementary dimension,

$$\sigma_1^b \sigma_{\lambda_{\alpha,\beta}} = \Big(\sum_{|\mu|=b} f_\mu \, \sigma_\mu \Big) \, \sigma_{\lambda_{\alpha,\beta}} = f_{\mu_{\alpha,\beta}} \; [\mathrm{pt.}] \, ,$$

where $\mu_{\alpha,\beta}$ is the complementary diagram to $\lambda_{\alpha,\beta}$ with respect to a $(k+1) \times (n-k)$ box (for example, in $\mathbb{G}(3, 8)$, $\mu_{3,2} = \ydiagram{5,4,1}$).

Set $f_{\alpha,\beta} := f_{\mu_{\alpha,\beta}}$.

Computing $f_{\alpha,\beta}$. $(k+1) \times (n-k)$ *box.* As a first case (that will be useful for computing $f_{\alpha,\beta}$), we compute the product of hooklengths of the $(k+1) \times (n-k)$ box.

The hooklengths are

n	$n-1$	$n-2$	$\dots$	$k+3$	$k+2$	$k+1$
$n-1$	$n-2$	$n-3$	$\dots$	$k+2$	$k+1$	k
	$\vdots$		$\ddots$		$\vdots$	
			$\dots$	4	3	2
$n-k$			$\dots$	3	2	1

The product of hooklengths in $(\ell-1)$st row from the bottom $(0 \le \ell - 1 \le k)$ is $\frac{(n-k+\ell)!}{\ell!}$, so the product of all hooklengths is

$$\prod_{\ell=0}^{k} \frac{(n-k+\ell)!}{\ell!}$$

and (since the diagram contains $(k+1)(n-k)$ boxes),

$$\frac{(\#\text{ boxes})!}{\text{Product of hooklengths}} = ((k+1)(n-k))! \prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!}.$$

$(k+1)\times(n-k)$ *box with a hook removed from the lower-right corner.* We illustrate the approach with a running example (with $k=4$, $n-k=6$, $\alpha=3$, $\beta=2$). The hooklengths are

10	9	7	6	5	2
9	8	6	5	4	1
7	6	4	3	2	
6	5	3	2	1	
2	1				

First, discard the boxes in the bottom row and the right column, obtaining

10	9	7	6	5
9	8	6	5	4
7	6	4	3	2
6	5	3	2	1

The resulting filling is nearly the filling of hooklengths in a $(k+1)\times(n-k)$ box. In fact, we can return to the latter case by adding a row and a column:

10	9	8	7	6	5
9	8	7	6	5	4
8	7	6	5	4	3
7	6	5	4	3	2
6	5	4	3	2	1

By the previous case, the product of hooklengths in this diagram is $\prod_{\ell=0}^{k} \frac{(n-k+\ell)!}{\ell!}$. It is now necessary to divide the above product by the entries in the added row and column.

The added row is $(\beta+1)$st in the diagram, so its product of hooklengths is $\frac{(n-k+\beta)!}{\beta!}$; the product of hooklengths in the added column (which is $(\alpha+1)$st) is $(\alpha+1)(\alpha+2)\cdots(\alpha+1+k) = \frac{(k+1+\alpha)!}{\alpha!}$. The row and the column intersect in a single box with hooklength $\alpha+\beta+1$, and, since it is divided out twice, it is necessary to multiply the result back by this number.

Finally, the two removed pieces of the diagram contribute

$$\begin{aligned} (n-k-\alpha-1)! \qquad & \text{(bottom row)} \\ (k+1-\beta-1)! = (k-\beta)! \qquad & \text{(right column)} \end{aligned}$$

The product of hooklengths of $\mu_{\alpha,\beta}$ is therefore

$$\left(\prod_{\ell=0}^{k} \frac{(n-k+\ell)!}{\ell!}\right) \frac{\alpha!}{(k+1+\alpha)!} \frac{\beta!}{(n-k+\beta)!} (\alpha+\beta+1)(n-k-\alpha-1)!(k-\beta)!$$

(For the example, this gives $118,513,704,960,000$, which agrees with what is obtained by simply multiplying out the hooklengths.)

We conclude that

(4.12)

$$f_{\alpha,\beta} = ((k+1)(n-k)-\alpha-\beta-1)! \left(\prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!}\right) \frac{(k+1+\alpha)!\,(n-k+\beta)!}{\alpha!\,\beta!\,(\alpha+\beta+1)\,(n-k-\alpha-1)!\,(k-\beta)!}$$

Note that this derivation is valid when $\alpha = 0$ or $\beta = 0$ (or both).

Returning to the number of lines computation. Adopt the convention that $f_{\alpha,\beta} = 0$ if $\alpha > n-k-1$ or $\beta > k$ (in these cases the hook does not fit into a $(k+1)\times(n-k)$ box).

The triple product $\sigma_a \sigma_1^b \sigma_{1^c}$ is equal to

$$\sigma_a \sigma_1^b \sigma_{1^c} = \begin{cases} \sigma_1^b(\sigma_{\lambda_{a,c-1}} + \sigma_{\lambda_{a-1,c}}) = (f_{a,c-1} + f_{a-1,c}), & a > 0,\ c > 0, \\ \sigma_a\, \sigma_1^b = \sigma_{\lambda_{a-1,0}} \sigma_1^b = f_{a-1,0}, & a > 0,\ c = 0, \\ \sigma_1^b\, \sigma_{1^c} = \sigma_1^b\, \sigma_{\lambda_{0,c-1}} = f_{0,c-1}, & a = 0, c > 0, \\ \sigma_1^b = ((k+1)(n-k))! \prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!}, & a = 0,\ c = 0 \end{cases}$$

(multiplied by the class of a point, in each of the four cases).

Neglecting the sign for the moment since it depends only on i and n, and adopting the convention that a sum indexed by the empty set is equal to 0,

$$\sum_{j=0}^{i}\binom{i}{j}\sigma_1\,\pi_*(H_1^j\,H_2^{f-i}\,H_3^{i-j}) = \sum_{j=k}^{i-n+k+1}\binom{i}{j}\,\sigma_{j-k}\sigma_1^{f-i+1}\sigma_{1^{i-j-n+k+1}}$$
$$= \sum_{a=0}^{i-n+1}\binom{i}{k+a}\,\sigma_a\sigma_1^{f-i+1}\sigma_{1^{i-n+1-a}}.$$

For $i < n-1$, the sum is empty. For $i = n-1$, the sum has only one term, which is equal to

$$\binom{n-1}{k}((k+1)(n-k))!\prod_{\ell=0}^{k}\frac{\ell!}{(n-k+\ell)!} \qquad \text{(in case } i = n-1\text{)}.$$

For $i > n-1$, we have

$$\begin{aligned}
&\sum_{a=0}^{i-n+1}\binom{i}{k+a}\,\sigma_a\sigma_1^{f-i+1}\sigma_{1^{i-n+1-a}}\\
&= \binom{i}{k}f_{0,i-n} + \left(\sum_{a=1}^{i-n}\binom{i}{k+a}\left(f_{a,i-n-a} + f_{a-1,i-n-(a-1)}\right)\right) + \binom{i}{k+i-n+1}f_{i-n,0}\\
&= \sum_{a=0}^{i-n}\left(\binom{i}{k+a} + \binom{i}{k+a+1}\right)f_{a,i-n-a}\\
&= \sum_{a=0}^{i-n}\binom{i+1}{k+a+1}f_{a,i-n-a}.
\end{aligned}$$

Since $f_{\alpha,\beta} = 0$ when $\alpha > n-k-1$, the upper bound of the summation can be refined to $M_i := \min(i-n,\, n-k-1)$. Similarly, since $f_{\alpha,\beta} = 0$ when $\beta > k$, we should have $i-n-a \le k$, so that $i-n-k \le a$; the lower bound of summation can then be refined to $m_i := \max(i-n-k,\, 0)$.

The region of summation is then the parallelogram below (the sum is taken over the integer points of the parallelogram).

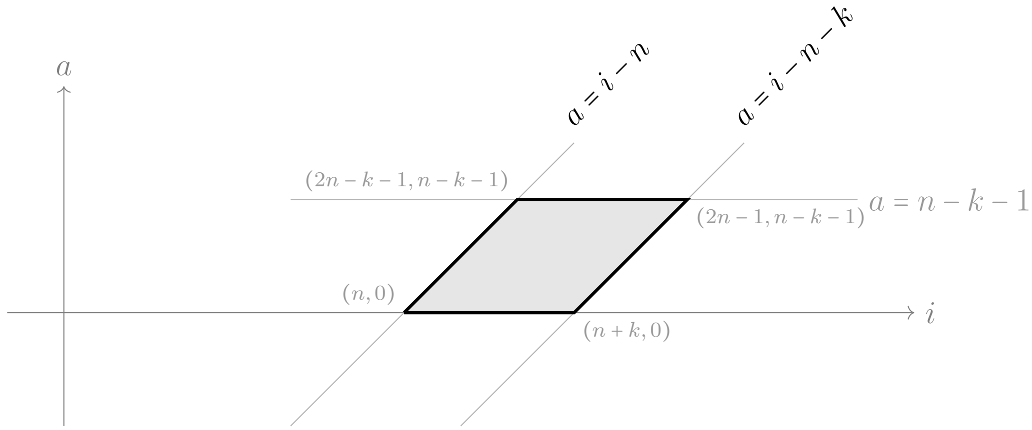


Figure 4.1: A typical parallelogram of summation.

The sum then becomes empty when $i-n-k > n-k-1$, i.e. when $i > 2n-1$. Comparing this to the previous upper bound on i, we note that $2n-1 \leq f$ if and only if $2n-1 \leq (k+1)(n-k)+n-2$, which in turn is equivalent to $1 \leq k(n-k-1)$. The latter inequality is false only when $k=0$ or $k=n-1$ (since $k \geq 0,\ n-k-1 \geq 0$), i.e. only for $\mathbb{G}(0,n) \cong \mathbb{P}^n$ and $\mathbb{G}(n-1,n) \cong \mathbb{P}^n$. (In these cases, the upper bound on the index was indeed $2n-2$, which is lower by 1).

Now, by (4.12) (note that "$\alpha+\beta+1$"$= (a)+(i-n-a)+1 = i-n+1$),

$$
\begin{aligned}
&\binom{i+1}{k+a+1} \frac{f_{a,i-n-a}}{((k+1)(n-k)-(i-n+1))!\left(\prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!}\right)} \\
&= \frac{(i+1)!}{(k+a+1)!\,(i-k-a)!}\ \frac{(k+a+1)!\,(i-k-a)!}{a!\,(i-n-a)!\,(i-n+1)\,(n-k-1-a)!\,(n+k-i+a)!} \\
&= \frac{(i+1)!}{a!}\ \frac{1}{(i-n-a)!}\ \frac{1}{i-n+1}\ \frac{1}{(n-k-1-a)!\,(n+k-i+a)!} \\
&= \binom{i+1}{a} \frac{(i+1-a)!}{(i-n-a)!}\ \frac{1}{i-n+1}\ \frac{1}{(n-k-1-a)!\,(n+k-i+a)!}
\end{aligned}
$$

Set

$$B_i := \begin{cases} \binom{n-1}{k}, & i = n-1, \\ \sum_{a=m_i}^{M_i} \binom{i+1}{a} \frac{(i+1-a)!}{(i-n-a)!} \frac{1}{(n-k-1-a)!\,(n+k-i+a)!}, & i > n-1, \end{cases}$$

where

$$m_i = \max(i-n-k,\, 0),$$
$$M_i = \min(i-n,\, n-k-1).$$

Noting that $(k+1)(n-k) - i + n - 1 = f - i + 1$, we get the following theorem.

Theorem 4.12. *Let $n \geq 0$, $0 < k < n-1$, $f = (k+1)(n-k)+n-2 = \dim(F_1(\mathbb{G}(k,n))$, $d \geq f-1$ be integers, and $|V| \subset \left|\mathcal{O}_{\mathbb{G}(k,n)}(d)\right|$ be a linear system of projective dimension $d+1-f$. If the locus of points of $\mathbb{F}l(k-1,k,k+1;n)$ where the morphism $V \otimes_k \mathcal{O}_{\mathbb{F}l(k-1,k,k;n)} \to \mathcal{P}^d_\phi(\pi^*\mathcal{O}_{\mathbb{G}}(d))$ does not have full rank is one-dimensional, then the number of lines that appear in $|V|$ is*

$$\prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!} \sum_{i=n-1}^{2n-1} (-1)^{i-n+1} \sigma(d+1,\, d+1-i) \binom{d+1-i}{f-i} B_i\, d^{f-i} \frac{(f-i+1)!}{i-n+1} \tag{4.13}$$

(In the statement of the theorem, we have adopted the convention that when $i = n-1$, the denominator of the expression $\frac{(f-i+1)!}{i-n+1}$ is equal to 1.)

Remark. Expression (4.13) can be rewritten as

$$\prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!} \sum_{i=n-1}^{2n-1} (-1)^{i-n+1} J(d,i) \binom{d+1-i}{f-i} B_i\, d^{f-i} \frac{(f-i+1)!}{i-n+1},$$

using the functions $J(n,k)$ introduced in §2.6.

Example. We check that in the case of a single hypersurface in $\mathbb{G}(1,3)$ the value of (4.13) agrees with the previously computed count of 1280 lines (§4.4.1).

The parameters are $k = 1$, $n = 3$, $f = (1+1)(3-1) + 3 - 2 = 5$, $d = f - 1 = 4$. The parallelogram of summation is

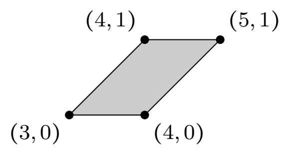


Figure 4.2: The parallelogram of summation for the case of a quartic hypersurface in $\mathbb{G}(1,3)$.

We compute that

$$
\begin{aligned}
B_2 &= \binom{2}{1} = 2.\\
B_3 &= \binom{3+1}{0}\frac{(3+1-0)!}{(3-3-0)!}\frac{1}{(3-1-1-0)!\,(3+1-3+0)!} = 1\cdot 4!\cdot 1\cdot\frac{1}{1!\cdot 1!} = 24.\\
B_4 &= \binom{4+1}{0}\frac{(4+1-0)!}{(4-3-0)!}\frac{1}{(3-1-1-0)!\,(3+1-4+0)!} + \binom{4+1}{1}\frac{(4+1-1)!}{(4-3-1)!}\frac{1}{(3-1-1-1)!\,(3+1-4+1)!}\\
&= 1\cdot\frac{5!}{1!}\cdot\frac{1}{0!\cdot 1!} + 5\cdot\frac{4!}{0!}\cdot\frac{1}{0!\cdot 1!} = 120 + 120 = 240.\\
B_5 &= \binom{5+1}{1}\frac{(5+1-1)!}{(5-4-1)!}\frac{1}{(3-1-1-1)!\,(3+1-5+1)!} = 6\cdot\frac{5!}{0!}\cdot\frac{1}{0!\cdot 0!} = 720.
\end{aligned}
$$

Also,

$$\prod_{\ell=0}^{k}\frac{\ell!}{(n-k+\ell)!} = \frac{0!}{2!}\cdot\frac{1!}{3!} = \frac{1}{12}.$$

Since $d+1 = f$ in the case of a single hypersurface, $\binom{d+1-i}{f-i} = 1$ for all i.

The relevant Stirling numbers of the first kind are

$$\sigma(5,5) = 1,\quad \sigma(5,4) = 10,\quad \sigma(5,3) = 35,\quad \sigma(5,2) = 50,\quad \sigma(5,1) = 24,$$

and $\sigma(5,k) = 0$ for other k.

The summation of (4.13) becomes

$$\frac{1}{12}\left(\sigma(5,3)\cdot 2\cdot 4^3\cdot\frac{4!}{1} - \sigma(5,2)\cdot 24\cdot 4^2\cdot\frac{3!}{1} + \sigma(5,1)\cdot 240\cdot 4^1\cdot\frac{2!}{2} - \sigma(5,0)\cdot 720\cdot 4^0\cdot\frac{1!}{3}\right) = 1280.$$

Duality. Because of the identification $\mathbb{G}(k,n) \cong \mathbb{G}(n-k-1,n)$ (together with the fact that this identification yields an isomorphism between $\mathcal{O}_{\mathbb{G}(k,n)}(1)$ and $\mathcal{O}_{\mathbb{G}(n-k-1,n)}(1)$), the value of the expression (4.13) should be invariant under the involution $k \leftrightarrow n-k-1$. Although this may not quite be apparent, it may be checked as follows.

Let $\Pi_{k,n}$ denote the set of integer points of the closed parallelogram with vertices

$$(n,\,0),\ (2n-k-1,\,n-k-1),\ (2n-1,\,n-k-1),\ (n+k,\,0)$$

(this is the region of summation found above). Let

$$\begin{aligned} p_{k,\,n,\,i,\,a} := \prod_{\ell=0}^{k} \frac{\ell!}{(n-k+\ell)!}\,(-1)^{i-n+1}\,\sigma(d+1,d+1-i)\binom{d+1-i}{f-i} d^{f-i}\frac{(f-i+1)!}{i-n+1} \times \cdots \\ \cdots \times \binom{i+1}{a}\frac{(i+1-a)!}{(i-n-a)!}\,\frac{1}{(n-k-1-a)!\,(n+k-i+a)!} \end{aligned}$$

Then the involution $(i,\,a) \leftrightarrow (i,\,i-a-n)$ of $\mathbb{Z}^2$ restricts to a bijection between $\Pi_{k,n}$ and $\Pi_{n-k-1,n}$, and $p_{k,\,n,\,i,\,a} = p_{n-k-1,\,n,\,i,\,i-n-a}$ (both facts are simple to check). Consequently, (4.13) is invariant under $k \leftrightarrow n-k-1$.

Comparison with the result for $\mathbb{P}^n$. In this paragraph we check that formally setting $k=0$ in formula (4.13) recovers formula (4.7) for projective space, with the following change: because $f < 2n-1$ in this case, the upper bound of the summation should be $i = 2n-2$ instead of $i = 2n-1$. When $k=0$, $f = 1(n) + n - 2 = 2n-2 = \dim(\mathbb{P}T_{\mathbb{P}^n})$. We obtain from (4.13)

$$\frac{0!}{n!}\sum_{i=n-1}^{2n-2}(-1)^{i-n-1}\,\sigma(d+1,d+1-i)\binom{d+1-i}{2n-2-i} B_i\, d^{2n-2-i}\,\frac{(2n-1-i)!}{i-n+1}.$$

Comparing with (4.7), we see the two formulas are equal if and only if

$$B_i\frac{(2n-1-i)!}{(i-n+1)\,n!} = \binom{i+1}{n} \qquad \text{for all } i = n-1,\ldots,2n-2.$$

For $i = n - 1$ (recalling that in this case the $i - n + 1$ in the denominator is by convention equal to 1), we have

$$B_i \frac{(2n-1-(n-1))!}{((n-1)-n+1)\,n!} = \binom{n-1}{0}\frac{n!}{n!} = 1 = \binom{n}{n}.$$

Suppose that $i \geq n$. The lower and upper bounds of summation are

$$m_i = \max(i-n,\, 0) = i-n, \qquad M_i = \min(i-n, n-1) = i-n,$$

since $i - n \leq n - 1$ if and only if $i \leq 2n - 1$. Thus $m_1 = M_i = i - n$ and each B_i has just one summand (geometrically, the parallelogram of summation "collapses" onto a diagonal side (side of slope one).) We compute that

$$B_i \frac{(2n-1-i)!}{(i-n+1)\,n!} = \left(\binom{i+1}{i-n}\frac{(n+1)!}{0!}\frac{1}{(2n-1-i)!\cdot 0!}\right)\frac{(2n-1-i)!}{(i-n+1)\,n!} = \binom{i+1}{n+1}\frac{n+1}{i-n+1} = \binom{i+1}{n},$$

so that (4.7) is indeed obtained.

The case $k = n - 1$ can be checked similarly (geometrically, this corresponds to the parallelogram of summation "collapsing" onto a horizontal side).

For $k = 0$ and $k = n - 1$, the Fano variety can be identified with a single-step flag variety, and its universal family with a two-step flag variety. The approach of this section can be modified to this set-up with few differences.

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